Hyperfine splitting of 2^3S_1 state in He^3

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Abstract

Relativistic corrections to the hyperfine splitting are calculated for the triplet 2^3S_1 state of the helium isotope He^3. High precision variational wave functions are employed, where the electron-electron correlations are well accounted for. Due to the unknown nuclear structure a comparison with the experimental result is performed via He^{3+} hyperfine splitting. Surprisingly, the second order contribution due to Fermi interaction, in spite of the additional small factor m/M, is very significant.

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The calculation of higher order relativistic and QED effects in few electron systems is a long standing problem. Various measurements of transition frequencies have reached the precision of few ppb, while no theoretical predictions are yet so accurate [1]. The usual approaches, which incorporate most of relativistic effects from the beginning, MCDF for example, are not capable to include electron-electron correlations in a complete way, thus their predictions are not sufficiently accurate. The alternative approach starts from the Schrödinger equation and incorporates relativistic effects perturbatively in the effective hamiltonian. The principal advantage is the simplicity and high accuracy of nonrelativistic wave functions, which allows for precise calculations of higher order relativistic effects. The cost one pays in the perturbative approach is the complexity and singularity of the effective hamiltonian. Historically, the helium fine structure splitting was the first application of the higher order effective (Breit-like) hamiltonian. Douglass and Kroll in [2] with the help of Bethe-Salpeter formalism derived effective operators at order $m\,\alpha^6$, which together with the second order correction from leading Breit terms gave a *complete* $m\,\alpha^6$ contribution to helium fine structure of $2P_J$ levels. It was the first disprove of the common believe, that Breit Hamiltonian could not be used at higher orders of perturbation calculus. 20 years later Khriplovich and coworkers [3] essentially rederived their result, without being aware of this former work, using the time ordered perturbation technique. With these effective operators they calculated fine structure splitting in positronium. The extension of the perturbative approach to S-states is a highly nontrivial task, see for example [4]. It is due to high singularity of effective operators, which matrix elements between S-states diverge, so one may question about the correctness of the perturbative approach. In spite of these divergences, it is possible to implement the perturbative approach to S-states as well. The idea lies in the regularization of Coulomb interaction, in such a way that all matrix elements become finite. All contribution coming from high-momenta are incorporated as local interaction, proportional to Dirac delta or it's derivatives. The coefficients are found by matching for example, the scattering amplitude and they depend on the regularization parameter. However, this dependence cancels out when all terms contributing to energy at specified order are summed up. As a test, one may rederive [4] Dirac energy levels at order $m\,\alpha^6$. This approach was successfully applied first in the calculation of positronium hfs [4]. This result was confirmed independently by 3 groups [5], two of them were applying directly Bethe-Salpeter equation, and this really serves as an independent check of this method. Little later, one performed [6] calculations of higher order relativistic corrections to helium 2^3S_1 level. The fact, that it is a triplet state, was a great simplification, since the wave function vanishes at $\mathbf{r}_1 = \mathbf{r}_2$. Very recently, Yelkhovsky [7] has derived complete set of operators for the ground 1^1S_0 state. The numerical calculation of their matrix elements is still under way. In this work we study He^3 hyperfine splitting, which was measured very precisely by Rosner and Pipkin in [8]. He^3 hfs is a nice example of difficulties in the bound state QED and perturbative approaches. The Fermi interaction, as given by Dirac $\delta^3(r)$ function is already quite singular. Incorporation of further relativistic effects leads to even more singular operators, which have to be properly handled. In the work [6] we have shown the practical way to handle singular effective operators for helium atom. Coulomb interaction is smoothed at small distances with the use of some parameter λ, to avoid small r nonintegrable singularities. The key point of this approach lies in the fact, that dependence on λ cancel out between all matrix elements, and correct energy levels are restored in the limit $\lambda \to \infty$. High electron momenta does not contribute here, because it is again triplet S-state, so one does not need to construct counter terms for matching scattering amplitude. Surprizingly, we have found a very large second order correction due to Fermi interaction. This correction requires a separate treatment and will be dealt with at the end of this work.

Hyperfine splitting in He^3 of 2^3S_1 state is due to the interaction of electron and helion (nucleus) magnetic moments. In the 2^3S_1 state both electron spins are parallel and sum to $S=1$ in contrast to the ground state were $S=0$. Magnetic moment of helion comes mainly from the neutron particle. It means that helion g-factor is negative and hyperfine sublevels are inverted with respect to hydrogen, the upper one has $S+I=1/2$ and the lower one $S+I=3/2$. Therefore, by a hyperfine splitting one means here $E_{\rm hfs} = E(1/2) - E(3/2)$. According to the perturbative approach the general expression for the hyperfine splitting up to the order $m\,\alpha^6$ is:

$$E_{\rm hfs} = \langle H^{(4)}_{\rm hfs}\rangle + \langle H^{(5)}_{\rm hfs}\rangle + \langle H^{(6)}_{\rm hfs}\rangle + 2\,\langle H^{(4)}\,\frac{1}{(E-H)'}\,H^{(4)}_{\rm hfs}\rangle + E_{\rm rec}\,. \tag{1}$$

$H^{(4)}_{\rm hfs}$ is:

$$H^{(4)}_{\rm hfs} = H^A_{\rm hfs} + H^B_{\rm hfs} + H^D_{\rm hfs}\,, \tag{2}$$

$$H^A_{\rm hfs} = \frac{8\,Z\,\alpha}{3\,m\,M}\biggl[\frac{\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_1}{4}\,\pi\,\delta^3(r_1) + \{1\to 2\}\biggr](1+k)\;(1+a)\,, \tag{3}$$

$$H^B_{\rm hfs} = \frac{Z\,\alpha}{2\,m\,M}\biggl[\frac{\boldsymbol{r}_1\times\boldsymbol{p}_1}{r_1^3} + \frac{\boldsymbol{r}_2\times\boldsymbol{p}_2}{r_2^3}\biggr]\cdot\boldsymbol{\sigma}_n\,(1+k)\,, \tag{4}$$

$$H^D_{\rm hfs} = -\frac{Z\,\alpha}{4\,m\,M}\biggl[\frac{\sigma^i_n\,\sigma^j_1}{r_1^3}\biggl(\delta^{ij} - 3\,\frac{r_1^i\,r_1^j}{r_1^2}\biggr) + \{1\to 2\}\biggr](1+k)\,, \tag{5}$$

where a, k are anomalous magnetic moments of the electron and the helion respectively. However, it is not commonly accepted such a notion for a nucleus. The relation of k with the magnetic moment of the nucleus with charge $Z\,e$ is $\boldsymbol{\mu} = 2\,(1+k)\,Z\,e/(2\,M)\,\boldsymbol{I}$. Masses m and M are of the electron and helion respectively. The expectation values of $H^B_{\rm hfs}$ and $H^D_{\rm hfs}$ vanish in 2^3S_1 state, but they contribute in the second order, last term in Eq. (1). $H^{(4)}$ is a Breit hamiltonian in the nonrecoil limit:

$$H^{(4)} = H^A + H^B + H^D\,, \tag{6}$$

$$H^A = -\frac{1}{8\,m^3}\,(p_1^4+p_2^4) + \frac{Z\,\alpha\,\pi}{2\,m^2}\,[\delta^3(r_1)+\delta^3(r_2)] - \frac{\alpha}{2\,m^2}\,p_1^i\biggl(\frac{\delta^{ij}}{r} + \frac{r^i\,r^j}{r^3}\biggr)\,p_2^j\,, \tag{7}$$

$$H^B = \biggl[\frac{Z\,\alpha}{4\,m^2}\biggl(\frac{\boldsymbol{r}_1\times\boldsymbol{p}_1}{r_1^3} + \frac{\boldsymbol{r}_2\times\boldsymbol{p}_2}{r_2^3}\biggr) - \frac{3\,\alpha}{4\,m^2}\,\frac{\boldsymbol{r}}{r^3}\times(\boldsymbol{p}_1-\boldsymbol{p}_2)\biggr]\,\frac{\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2}{2}\,, \tag{8}$$

$$H^D = \frac{\alpha}{4\,m^2}\,\frac{\sigma_1^i\,\sigma_2^j}{r^3}\biggl(\delta^{ij} - \frac{3\,r^i\,r^j}{r^2}\biggr)\,, \tag{9}$$

where $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ and $r = |\boldsymbol{r}|$. Any possible recoil corrections at order $m\,\alpha^6$ are included in $E_{\rm rec}$. In spite of the small additional factor m/M their contribution is significant. This effect [15] could be associated to the mixing of 2^3S_1 and 2^1S_1 states caused by the Fermi interaction. $\langle H^{(5)}_{\rm hfs}\rangle$ is delta-like term with the coefficient given by the two-photon forward scattering amplitude. It is the same like in hydrogen and strongly depends on the nuclear structure. It also automatically includes nuclear recoil effects and inelastic contribution (nuclear polarizability). $\langle H^{(5)}_{\rm hfs}\rangle$ has been considered in detail for the case of hydrogen and muonic hydrogen. However, there are no sufficient experimental data available for helion, He^3 nucleus, therefore we were unable to estimate these contributions. Moreover, it would be incorrect to apply this correction for point-like nucleus, because high energy photon momenta are involved, where nucleus, could not be approximated as a point like particle. Therefore we will leave this contribution unevaluated, and at the end for the comparison with an experiment, will subtract the appropriately scaled hydrogenic value for hyperfine splitting The last term $H^{(6)}_{\rm hfs}$ includes spin dependent operators, which contribute at order $m\,\alpha^6$. It is only this term, which derivation was not performed so far in the literature and is presented here. The detailed description on derivation of effective hamiltonian, reader may find in former works, for example in [4]. There are four time ordered diagrams, which contributes to $H^{(6)}_{\rm hfs}$ and are presented in Fig 1. The first two are the same as in hydrogen, other two are essentially three body terms. One derives the following expressions, which corresponds to these diagrams.

$$
\begin{aligned}
H^{(6)}_{\rm hfs} &= V_1+V_2+V_3+V_4\,, &(10)\\
V_1 &= -(1+k)\,\frac{\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_1}{48\,M\,m^3}\Bigl\{2\,p_1^2\,4\,\pi\,Z\,\alpha\,\delta^3(r_1)+2\cdot 4\,\pi\,Z\,\alpha\,\delta^3(r_1)\,p_1^2+\Bigl[p_1^2,\Bigl[p_1^2,\frac{Z\,\alpha}{r_1}\Bigr]\Bigr]\Bigr\}\\
&\quad+\{1\to 2\}\,, &(11)\\
V_2 &= (1+k)\,\frac{(Z\,\alpha)^2}{r_1^4}\,\frac{\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_1}{6\,M\,m^2}+\{1\to 2\}\,, &(12)\\
V_3 &= -(1+k)\,\frac{\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_1}{6\,M\,m^2}\,\frac{Z\,\alpha\,\boldsymbol{r}_1}{r_1^3}\cdot\frac{\alpha\,\boldsymbol{r}}{r^3}+\{1\leftrightarrow 2\}\,, &(13)\\
V_4 &= -(1+k)\,\frac{\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_2}{6\,M\,m^2}\,\frac{Z\,\alpha\,\boldsymbol{r}_1}{r_1^3}\cdot\frac{\alpha\,\boldsymbol{r}}{r^3}+\{1\leftrightarrow 2\}\,. &(14)
\end{aligned}
$$

It is worth noting that V_1 ans V_2 are the same as in hydrogen. Matrix element of $H^{(6)}_{\rm hfs}$ and last term in Eq. (1) are separately divergent at small r_1 and r_2. However these divergences cancel out in the sum. It is because the high electron momentum contribution is absent in the nonrecoil limit. For hydrogen this sum is equal to the expectation value of $\gamma\,A$ on Dirac wave function. We introduce now the following regulator λ to the electron-nucleus Coulomb interaction

$$
\frac{Z\,\alpha}{r_i}\to\frac{Z\,\alpha}{r_i}\,(1-e^{-\lambda\,m\,Z\,\alpha\,r_i})\,, \tag{15}
$$

in all hamiltonians in Eq.(1), as well as in the nonrelativistic one. This leads to the following further replacements in $H^{(6)}_{\rm hfs}$

$$
4\,\pi\,Z\,\alpha\,\delta^3(r_i)\equiv-\nabla^2\,\frac{Z\,\alpha}{r_i}\to-\nabla^2\,\frac{Z\,\alpha}{r_i}(1-e^{-\lambda\,m\,Z\,\alpha\,r_i})\,, \tag{16}
$$

$$
\frac{(Z\,\alpha)^2}{r_i^4}\equiv\Bigl(\nabla\frac{Z\,\alpha}{r_i}\Bigr)^2\to\Bigl(\nabla\frac{Z\,\alpha}{r_i}(1-e^{-\lambda\,m\,Z\,\alpha\,r_i})\Bigr)^2\,. \tag{17}
$$

Once the interaction is regularized, one can calculate all matrix elements and take the limit $\lambda\to\infty$. As a first step, using formulas from [4], we rederived the known relativistic correction to hfs in hydrogen

$$\delta E_{\rm hfs} = (1+k)\,\frac{\mu^3}{m\,M}\,\frac{(Z\,\alpha)^6}{n^3}\,\frac{\boldsymbol{\sigma}_n\boldsymbol{\sigma}_e}{4}\left(\frac{44}{9}+\frac{4}{n}-\frac{44}{9\,n^2}\right), \tag{18}$$

where n is a principal quantum number. It agrees with that, obtained directly from the Dirac equation. Since for helium, all matrix elements could be calculated only numerically, we will transform effective operators to the regular form, where λ could be taken to infinity before the numerical calculations. The initial expression for a complete set of relativistic corrections in atomic units is (with implicit λ regularization and excluding recoil):

$$\delta E_{\rm hfs} = |1+k|\,\frac{\mu^3}{m\,M}\,\alpha^6\,\mathcal{E}\,, \tag{19}$$

$$\mathcal{E} = \mathcal{E}_A+\mathcal{E}_B+\mathcal{E}_D+\mathcal{E}_N\,, \tag{20}$$

$$\begin{aligned}\mathcal{E}_A = 2\Big\langle&\Big\{-\frac{1}{8}\,(p_1^4+p_2^4)+\frac{Z\,\pi}{2}\,[\delta^3(r_1)+\delta^3(r_2)]-\frac{1}{2}\,p_1^i\left(\frac{\delta^{ij}}{r}+\frac{r^i\,r^j}{r^3}\right)\Big\}\\ &\frac{1}{(E-H)'}\,2\,Z\,\pi\,[\delta^3(r_1)+\delta^3(r_2)]\Big\rangle,\end{aligned} \tag{21}$$

$$\begin{aligned}\mathcal{E}_B = 2\Big\langle&\Big\{\frac{Z}{4}\Big[\frac{\boldsymbol{r}_1\times\boldsymbol{p}_1}{r_1^3}+\frac{\boldsymbol{r}_2\times\boldsymbol{p}_2}{r_2^3}\Big]-\frac{3}{4}\,\frac{\boldsymbol{r}\times(\boldsymbol{p}_1-\boldsymbol{p}_2)}{r^3}\Big\}\frac{1}{(E-H)'}\\ &\frac{Z}{2}\,\Big[\frac{\boldsymbol{r}_1\times\boldsymbol{p}_1}{r_1^3}+\frac{\boldsymbol{r}_2\times\boldsymbol{p}_2}{r_2^3}\Big]\Big\rangle,\end{aligned} \tag{22}$$

$$\mathcal{E}_D = 2\Big\langle\frac{1}{4}\left(\frac{\delta^{ij}}{r^3}-3\,\frac{r^i\,r^j}{r^5}\right)\frac{1}{(E-H)'}\left(-\frac{Z}{4}\right)\Big[\left(\frac{\delta^{ij}}{r_1^3}-3\,\frac{r_1^i\,r_1^j}{r_1^5}\right)+\left(\frac{\delta^{ij}}{r_2^3}-3\,\frac{r_2^i\,r_2^j}{r_2^5}\right)\Big]\Big\rangle, \tag{23}$$

$$\begin{aligned}\mathcal{E}_N = \Big\langle&-\frac{1}{4}\left(p_1^2\,\delta^3(r_1)+p_2^2\,\delta^3(r_2)\right)-\frac{1}{16}\left(\Big[p_1^2,\Big[p_1^2,\frac{Z}{r_1}\Big]\Big]+\Big[p_2^2,\Big[p_2^2,\frac{Z}{r_2}\Big]\Big]\right)\\ &+\frac{1}{2}\left(\frac{Z^2}{r_1^4}+\frac{Z^2}{r_2^4}\right)-\frac{\boldsymbol{r}}{r^3}\left(\frac{Z\,\boldsymbol{r}_1}{r_1^3}-\frac{Z\,\boldsymbol{r}_2}{r_2^3}\right)\Big\rangle,\end{aligned} \tag{24}$$

where $\mathcal{E}_N = \langle H^{(6)}_{\rm hfs}\rangle$, and we used the following formulas for hfs of $^3S_1$ states:

$$\langle\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_1\rangle = \langle\boldsymbol{\sigma}_n\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)/2\rangle = -3\,, \tag{25}$$

$$\langle\sigma_1^i\,\sigma_2^j\,Q_1^{ij}\;\sigma_n^a\,(\sigma_1+\sigma_2)^b\,Q_2^{ab}\rangle = -2\,Q_1^{ij}\,Q_2^{ij}\,, \tag{26}$$

for symmetric and traceless Q^{ij}. There is also a one loop radiative correction

$$\mathcal{E}_R = 2\,Z^2\left(\ln 2-\frac{5}{2}\right)\langle\pi\,\delta^3(r_1)+\pi\,\delta^3(r_2)\rangle\,, \tag{27}$$

which is similar to that in hydrogen. It will not contribute to the special difference between the helium and hydrogen-like helium hfs, therefore we will not consider it any further. The initial expression is rewritten to the regular form, where λ regularization is not necessary. The operators in second order terms $\mathcal{E}_A$ are transformed with the use of

$$H'^A \equiv H^A-\frac{1}{4}\left(\frac{Z}{r_1}+\frac{Z}{r_2}\right)(E-H)-\frac{1}{4}\,(E-H)\left(\frac{Z}{r_1}+\frac{Z}{r_2}\right), \tag{28}$$

$$\begin{aligned}4\,\pi\,Z\,[\delta^3(r_1)+\delta^3(r_2)]' \equiv\;& 4\,\pi\,Z\,[\delta^3(r_1)+\delta^3(r_2)]\\ &+2\left(\frac{Z}{r_1}+\frac{Z}{r_2}\right)(E-H)+2\,(E-H)\left(\frac{Z}{r_1}+\frac{Z}{r_2}\right).\end{aligned} \tag{29}$$

This transformation leads to new form for $\mathcal{E}'_A$ and $\mathcal{E}'_N$, such that

$$\mathcal{E}_A + \mathcal{E}_N = \mathcal{E}'_A + \mathcal{E}'_N\,, \tag{30}$$

$$\mathcal{E}'_A = 2\left\langle H'^A \frac{1}{(E-H)'}\, 2\,Z\,\pi\,[\delta^3(r_1)+\delta^3(r_2)]'\right\rangle, \tag{31}$$

$$\begin{aligned}\mathcal{E}'_N = \Big\langle & \left(E-\frac{1}{r}\right)^2\left(\frac{Z}{r_1}+\frac{Z}{r_2}\right)+\left(E-\frac{1}{r}\right)\left(\frac{Z^2}{r_1^2}+\frac{Z^2}{r_2^2}+4\,\frac{Z}{r_1}\frac{Z}{r_2}\right)+2\,\frac{Z}{r_1}\frac{Z}{r_2}\left(\frac{Z}{r_1}+\frac{Z}{r_2}\right)\\ & -\left(E-\frac{1}{r}+\frac{Z}{r_2}-\frac{p_2^2}{2}\right)4\,\pi\,Z\,\delta^3(r_1)-\frac{5\,Z}{4}\,\frac{r^i}{r^3}\left(\frac{r_1^i}{r_1^3}-\frac{r_2^i}{r_2^3}\right)\\ & +p_1^i\,\frac{Z^2}{r_1^2}\,p_1^i-p_2^2\,\frac{Z}{r_1}\,p_1^2+2\,p_2^i\,\frac{Z}{r_1}\left(\frac{\delta^{ij}}{r}+\frac{r^i\,r^j}{r^3}\right)p_1^j\Big\rangle\\ & -\frac{1}{4}\left\langle\frac{Z}{r_1}+\frac{Z}{r_2}\right\rangle\langle 4\,\pi\,Z\,(\delta^3(r_1)+\delta^3(r_2))\rangle+2\left\langle\frac{Z}{r_1}+\frac{Z}{r_2}\right\rangle\langle H^A\rangle\,.\end{aligned} \tag{32}$$

In the numerical calculations of these matrix elements we follow the approach developed by Korobov [9]. The S wave function is expanded in the sum of pure exponentials

$$\phi = \sum_{i=1}^{N} v_i\,(e^{-\alpha_i\,r_1-\beta_i\,r_2-\gamma_i\,r}-(r_1\leftrightarrow r_2))\,, \tag{33}$$

with randomly chosen $\alpha_i, \beta_i, \gamma_i$ in some specified limits. This basis set has been proven to give excellent results for the nonrelativistic energy and the wave function. Moreover, its simplicity allows for the calculations of relativistic corrections. With basis set $N=1200$ we obtained the nonrelativistic energy (without the mass polarization term $\boldsymbol{p}_1\boldsymbol{p}_2\,\mu/M$)

$$E = -2.17522937823679130\,57(1)\,, \tag{34}$$

slightly below the previous result in [10]. Expectation values of Dirac delta function without and with the mass polarization term are correspondingly:

$$\langle\, 4\,\pi\,(\delta^3(r_1)+\delta^3(r_2))\rangle = 33.184142630(1)\,, \tag{35}$$

$$\langle\, 4\,\pi\,(\delta^3(r_1)+\delta^3(r_2))\rangle_{\rm MP} = 33.184152589(1)\,. \tag{36}$$

The last one gives the leading hfs in helium, which is

$$E_{\rm hfs} = 2\,Z\,\alpha^4\,\frac{\mu^3}{m\,M}\,|1+k|\,(1+a)\,\langle\pi\,(\delta^3(r_1)+\delta^3(r_2))\rangle_{\rm MP} \approx 6\,740\,451\,{\rm kHz}\,, \tag{37}$$

where we use values of physical constants from Ref. [11] with one exception [12]. Numerical results for $\mathcal{E}_X$ with $X=A,B,D,N$ are presented in Table I. The inversion of $H-E$ in $\mathcal{E}_A$ is performed in the similar basis set as for 2^3S_1 wave function, however the nonlinear parameters had have to be properly chosen, to obtain a sufficiently accurate result. Namely, if $0<X,Y,Z<1$ are independent pseudo random numbers with homogeneous distribution, then:

$$\alpha = A_2\,X^{-n}+A_1\,, \tag{38}$$

$$\beta = (B_2-B_1)\,Y+B_1\,, \tag{39}$$

$$\gamma = (C_2-C_1)\,Z+C_1\,. \tag{40}$$

Parameters A, B, C and n are found, by minimization of the second order term with regularized Dirac delta on both sides. The inversion of $H-E$ in $\mathcal{E}_B$ is performed in the basis set of the form

$$\boldsymbol{\phi} = \boldsymbol{r}_1 \times \boldsymbol{r}_2 \sum_{i=1}^{N} v_i \left(e^{-\alpha_i r_1 - \beta_i r_2 - \gamma_i r} + (r_1 \leftrightarrow r_2)\right). \tag{41}$$

Unfortunately, we have not been able to get a reliable number for $\mathcal{E}_D$ with this numerical approach. The reason is that operators in $\mathcal{E}_D$ are so singular, that this basis set gives a very pure convergence. The result presented in Table I, is obtained analytically within $1/Z$ approximation, namely we neglected completely electron-electron interaction and corrected this value by factor $\langle \pi(\delta^3(r_1) + \delta^3(r_2))\rangle/9$. The estimated uncertainty is of the order of 10%. The total contribution of $m\,\alpha^6$ term (without recoil $E_{\rm rec}$) to hfs is

$$E^{(6)} = |1+k|\,\frac{\mu^3}{m\,M}\,\alpha^6\,200.9844(5) = 2171.440(5)\,\text{kHz}, \tag{42}$$

what could compared to the leading Fermi contact interaction in Eq. (37), $E^{(6)}/E_{\rm hfs} \approx 0.000322$. Between all the contributions to $\mathrm{He}^3(2^3S_1)$ hyperfine splitting in Eq.(1), $H^{(5)}$, essentially the nuclear structure contribution, requires input from the nuclear physics to be reliable evaluated. This is the reason we do not present final theoretical predictions for hfs, to compare with the precise measurement in [8]

$$E_{\rm hfs}(\mathrm{He}) = 6\,739701.177(16)\,\text{kHz}. \tag{43}$$

Instead, we can compare our result indirectly by subtracting the ground state hfs of helium ion as measured in [13]

$$E_{\rm hfs}(\mathrm{He}^+) = 8\,665\,649.867(10)\,\text{kHz}, \tag{44}$$

by composing the following difference

$$\Delta E_{exp} = E_{\rm hfs}(\mathrm{He}) - \frac{3}{4}\,\frac{\langle \pi\,(\delta^3(r_1) + \delta^3(r_2))\rangle_{\rm MP}}{8}\,E_{\rm hfs}(\mathrm{He}^+) = -38.998(19)\,\text{kHz}. \tag{45}$$

In this way, nuclear structure contribution, of order $m\,\alpha^5$ cancels out, as well as the leading Fermi contact interaction. What remain are electron–electron correlation effects. Theoretical predictions for this difference are (see Table I)

$$\Delta E_{rel} = |1+k|\,\frac{\mu^3}{m\,M}\,\alpha^6\,1.8795(5) = 20.306(5)\,\text{kHz}. \tag{46}$$

We do not associate here the uncertainty due to higher order terms. A disagreement of the theoretical result in Eq. (46) with the experimental one in Eq. (45) indicates that the presented calculation is incomplete. However the old calculations of relativistic corrections in [14] using an approximate DHF wave function, led to the result which is in an agreement with experiment $\Delta E_{\rm old} = -32(22)\,\text{kHz}$. This agreement is only illusive, since there is a correction discovered by Sternheim [15], which is denoted here by $E_{\rm rec}$. It is the second order contribution due to Fermi interaction $H^A_{\rm hfs}$ in Eq. (3)

$$E_{\rm rec} = \langle\, H^A_{\rm hfs}\, \frac{1}{(E-H)'}\, H^A_{\rm hfs}\,\rangle\,. \tag{47}$$

It could be understood as a recoil correction since it includes additional small factor m/M. Sternheim noticed that $H^A_{\rm hfs}$ mixes 2^3S_1 and 2^1S_1 states, what together with small energy difference between these states leads to strong enhancement of this recoil correction. Sternheim result which includes only 2^1S_1 intermediate state is $\delta E = -66.7(3)$ kHz, what nicely would explain the discrepancy of 59.3 kHz. However, the inclusion of higher excited states, will lead to the infinite result. Moreover, this correction is partially included in $\langle H^{(5)}_{\rm hfs}\rangle$, and only after proper subtraction it becomes finite. Hopefully, all divergence or in other words dependence on regularization parameter λ cancel out in this particular difference between helium atom and helium ion hyperfine splitting. Therefore instead of Eq. (47) we calculate the difference

$$\begin{aligned}\Delta E_{\rm rec} &= E_{\rm rec}({\rm He}) - \frac{3}{4}\,\frac{\langle\pi\,(\delta^3(r_1)+\delta^3(r_2))\rangle_{\rm He}}{8}\,E_{\rm rec}({\rm He}^+) = \frac{3}{2}\left(\frac{Z\,\alpha\,(1+k)}{3\,m\,M}\right)^2\\ &\times\biggl\{\frac{1}{2}\left\langle 4\,\pi\,[\delta^3(r_1)+\delta^3(r_2)]\,\frac{1}{(E-H)'}\,4\,\pi\,[\delta^3(r_1)+\delta^3(r_2)]\right\rangle\\ &+\frac{1}{2}\left\langle 4\,\pi\,[\delta^3(r_1)-\delta^3(r_2)]\,\frac{1}{E-H}\,4\,\pi\,[\delta^3(r_1)-\delta^3(r_2)]\right\rangle\\ &-\frac{\langle\pi\,(\delta^3(r_1)+\delta^3(r_2))\rangle_{\rm He}}{8}\left\langle 4\,\pi\,\delta^3(r)\,\frac{1}{(E-H)'}\,4\,\pi\,\delta^3(r)\right\rangle_{{\rm He}^+}\biggr\}\end{aligned} \tag{48}$$

Further evaluations proceed in the similar way as for $\mathcal{E}_A$. With the help of Eq.(29) one subtracts algebraically all divergences and the remaining finite terms are calculated numerically. The result is

$$\Delta E_{\rm rec} = -\frac{3}{2}\left(\frac{Z\,\alpha\,(1+k)}{3\,m\,M}\right)^2 m^5\,\alpha^4\,14491.0(1) = -|1+k|\,\frac{m^2}{M}\,\alpha^6\,5.5971 = -60.471\,{\rm kHz} \tag{49}$$

The complete theoretical predictions are the sum of Eqs. (46) and (49)

$$\Delta E_{\rm th} = -|1+k|\,\frac{\mu^3}{m\,M}\,\alpha^6\,3.71756 = -40.165\,{\rm kHz} \tag{50}$$

It nicely agrees with the experimental value –38.998 kHz from Eq. (45). The small difference of 1.167 kHz could be associated to higher order QED corrections. In summary we have calculated relativistic and a dominating recoil correction to helium hyperfine splitting. Due to unknown nuclear structure we considered such a difference of hfs of helium atom and He$^+$ ion, that cancels out this nuclear structure contribution. The recoil correction, namely the second order term in the Fermi interaction give a significant contribution to hfs, due to mixing of singlet and triplet S-states. Was this correction included, a good agreement with experiment is achieved.

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REFERENCES

[1] G.W.F. Drake and W.C. Martin, Can. J. Phys. **76**, 679 (1998).
[2] M. Douglass and N.M. Kroll, Ann. Phys. (N.Y) **82**, 89 (1974).
[3] I.B. Khriplovich, A.I. Milstein, A.S. Yelkhovsky, Phys. Rev. Lett. **77**, 4323 (1993)
[4] K. Pachucki, Phys. Rev. A 56, 297-304 (1997).
[5] G. Adkins and J. Sapirstein, Phys. Rev. A **58**, 3552 (1998), Erratum: Phys. Rev. A 61, 069902 (2000); A. Czarnecki, K. Melnikov, and A. Yelkhovsky, Phys. Rev. Lett. **82** 311 (1999); A.P. Burichenko, hep-ph/0004063.
[6] K. Pachucki, Phys. Rev. Lett. **84**, 4561 (2000).
[7] A. Yelkhovsky, hep-ph/0103241.
[8] S.D. Rosner and F.M. Pipkin, Phys. Rev. A, **1**, 571 (1970).
[9] V.I. Korobov, Phys. Rev. A **61**, 064503 (2000).
[10] G.W.F. Drake and Z. C. Yan, Chem. Phys. Lett. 229, 486 (1994).
[11] P.J. Mohr and B.N. Taylor, Rev. Mod. Phys. **72**, 351 (2000).
[12] The experimental value for the helion magnetic moment is somehow problematic. The reference [11] presents values for the screened helion in He^3, which involves unknown binding corrections. Therefore we have decided to follow the reference [8] and adopt the value $\mu_h/\mu_p = -0.761\,812\,0(7)$ for the helion–proton magnetic moment ratio.
[13] H.A. Schuessler, E.N. Fortson, and H.G. Dehmelt, Phys. Rev. **187**, 5 (1969).
[14] A.M. Sessler and H.M. Foley, Phys. Rev. **98**, 6 (1955).
[15] M.M. Sternheim, Phys. Rev. Lett. **15**, 545, (1965).

FIGURES

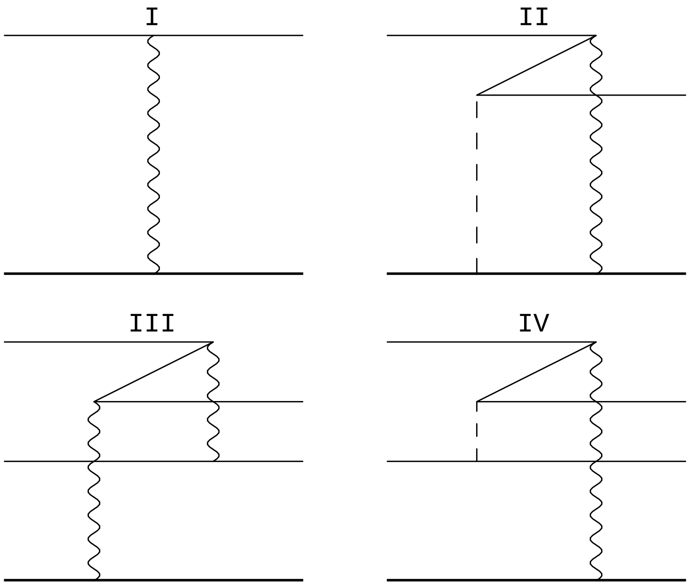

FIG. 1. Time ordered diagrams contributing to helium hyperfine structure at order $m\,\alpha^6$. Dashed line is a Coulomb photon, the wavy line is the transverse photon, the thicker vertical line denotes nucleus, two other electrons.

TABLES

contribution	$\vert 1+k\vert\ m^2/M\,\alpha^6$
$\mathcal{E}_A$	202.6761
$\mathcal{E}_B$	0.0059
$\mathcal{E}_D$	0.0054(5)
$\mathcal{E}_N$	-1.7030
$\mathcal{E}$	200.9844(5)
$24\,\langle\,\pi(\delta^3(r_1)+\delta^3(r_2))\rangle$	199.1049
$\Delta\mathcal{E}_{\rm rel}$	1.8795(5)
$\Delta\mathcal{E}_{\rm rec}$	-5.5971
$\Delta\mathcal{E}_{\rm th}=\Delta\mathcal{E}_{\rm rel}+\Delta\mathcal{E}_{\rm rec}$	-3.7176(5)
$\Delta\mathcal{E}_{\rm exp}$	-3.6095(18)

TABLE I. Numerical results for contributions at order $m\,\alpha^6$ to helium hyperfine structure. The factor 24 in the above comes from Breit correction Eq.(18) with $n=1$ times 3/4 from spin algebra times $Z^6/8$.