Trautman–Bondi mass and angular momentum for scalar field and gravity

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Abstract. The energy and angular momentum at null infinity are presented with the help of a simple example of a massless scalar field in Minkowski spacetime. It turns out that the case of the massless scalar field in Minkowski spacetime already exhibits all the essential features of the problem at hand, while avoiding various technicalities which arise when one wishes to describe Einstein gravity. In General Relativity we present a new variational formulation on hypersurfaces which are space-like inside and light-like near future null infinity. The formulae, we obtain, correspond to the mass loss formula.

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1 Scalar field on a hyperboloid

Starting from the Lagrangian density $L := -\frac{1}{2}\sqrt{-\det \eta_{\mu\nu}}\eta^{\mu\nu}\partial_\nu\phi\partial_\mu\phi$ for a scalar field ϕ in a flat Minkowski space M with the metric $\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \rho^{-2}\big(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2 + \frac{2\mathrm{d}s\mathrm{d}\rho}{\sqrt{1+\rho^2}} + \frac{\mathrm{d}\rho^2}{1+\rho^2} - \rho^2\mathrm{d}s^2\big)$ one can consider (see [5]) the variational formula on a hyperboloid $\Sigma := \{x \in M \mid x^0 := s = \text{const.}\}$:

$$-\delta\mathcal{H} = \int_\Sigma \left(\dot{\pi}\delta\psi - \dot{\psi}\delta\pi\right) + \int_{\partial\Sigma} \sin\theta\dot{\psi}\delta\psi \tag{1}$$

where $\mathcal{H} := \int_\Sigma H$, the density of the Hamiltonian in terms of conformally rescaled phase variables (π, ψ) takes the form

$$H(\pi,\psi) := \frac{\Omega^2}{2\rho^2}\sin\theta\left[\left(\rho\psi_{,\rho}\right)^2 + \big(\frac{\pi\rho^2\sqrt{1+\rho^2}}{\Omega^2\sin\theta} + \psi_{,\rho}\big)^2 + \overset{\circ}{\gamma}{}^{AB}\psi_{,A}\psi_{,B} - \frac{\Omega^2}{6\rho^2}R(g)\psi^2\right]$$

and $\overset{\circ}{\gamma}_{AB}\mathrm{d}x^A\mathrm{d}x^B = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$, $(A = \theta, \varphi)$, $\psi := \Omega^{-1}\phi$[a].
Let $\overline{M}$ denote the standard conformal completion at future null infinity $\mathscr{I}^+$ of M, let $\overline{\Sigma}_\tau$ be the closure[b] of Σ_τ in $\overline{M}$, set $S_\tau := S(\tau, \rho = 0) = \partial\overline{\Sigma}_\tau = \overline{\Sigma}_\tau \cap \mathscr{I}^+$. The

[a] Ω is a conformal factor such that $\rho^{-1}\Omega = O(1)$ and $\psi = O(1) = \partial_\rho\psi$.
[b] Σ_τ is a hyperboloid at “retarded time” $s = \tau$.

reader not familiar with the notion of Scri can simply think of the S_τ's as "spheres at infinity" on the hypersurfaces Σ_τ. ($\mathscr{I}^+ = \{x \in \overline{M} \mid \rho = 0\}$.)

Variational formula (1) describes an opened Hamiltonian system because we are not allowed to kill the boundary term and our Hamiltonian $\mathcal{H}$ is not conserved in time $-\partial_0 \mathcal{H} = \int_{S_\tau} \sin\theta(\dot\psi)^2$. Similarily, for the angular momentum along z-axis (which is related to the vector field $X_J = \partial_\varphi$) we obtain non-conservation law $-\partial_0 J_z = \int_{S_{\tau_0}} \sin\theta\dot\psi\psi_{,\varphi}$ where $J_z := \int_\Sigma \pi\psi_{,\varphi}$.

For $\tau > \tau_0$ let $N_{[\tau_0,\tau]} := \cup_{u\in[\tau_0,\tau]} S_u$, so $N_{[\tau_0,\tau]}$ be a null hypersurface contained in $\mathscr{I}^+$ with boundary $S_\tau \cup S_{\tau_0}$. An attempt to treat separately the hyperboloid and Scri leads to various difficulties in the Hamiltonian approach. However, the ADM energy assigned to the hyperboloid Σ_τ plus $N_{]-\infty,\tau]}$ – a piece of Scri between hyperboloid and spatial infinity (cf. [3]), enables one to remove an infinite tail $N_{]-\infty,\tau_0]}$ and apply the remaining Trautman-Bondi energy as a Hamiltonian. More precisely, we can consider a Hamiltonian system on a surface $\Sigma_\tau \cup N_{[\tau,\tau_0]}$ and according to [5] we have the following variational formula

$$-\delta m_{TB} = \int_{\Sigma_\tau \cup N_{[\tau,\tau_0]}} \left(\dot\pi\delta\psi - \dot\psi\delta\pi\right) + \int_{S_{\tau_0}} \pi\delta\psi \tag{2}$$

where $m_{TB} := \int_{\Sigma_\tau \cup N_{[\tau,\tau_0]}} H = \int_{\Sigma_{\tau_0}} H$ is the Trautman-Bondi energy at retarded time τ_0, the density of the Hamiltonian on N is defined by $H := \pi\psi_{,u}$ and $\pi|_N = \sin\theta\dot\psi$. Killing the term at S_{τ_0} in (2) by an appropriate choice of boundary conditions, our system (2) becomes Hamiltonian as a usual infinite dimensional dynamical system. This can be achieved, assuming for example that $\delta\psi|_{S_{\tau_0}} = 0$, which simply means that ψ is fixed at the time τ_0. The precise meaning of those heuristic considerations will be given in [6].

2 General Relativity

The curved space-time M equipped with a pseudoriemannian metric of the form

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\frac{V}{r}\mathrm{e}^{2\beta}\mathrm{d}u^2 - 2\mathrm{e}^{2\beta}\mathrm{d}u\mathrm{d}r + r^2\gamma_{AB}(\mathrm{d}x^A - U^A\mathrm{d}u)(\mathrm{d}x^B - U^B\mathrm{d}u) \tag{3}$$

enables one to consider the initial value problem on a light-like hypersurface $C := \{x \in M \mid x^0 = u = \text{const.},\ r \ge r_0\}$ with the boundary $\partial C = S(u, r = r_0) \cup S(u, r = \infty)$, where $S(u, r = r_0) := \{x \in M \mid x^0 = u = \text{const.},\ r = r_0\}$ and $S(u, r = \infty) = S_u \subset \mathscr{I}^+$ is a sphere at the future null infinity. We also assume that $\sqrt{\det\gamma_{AB}} = \sin\theta$. It has been shown in [5] how to compose generating formulae on a space-like and null hypersurfaces along two-dimensional boundary, where they meet. More precisely, let O be a space-like hypersurface with $\partial O = S(u, r_0)$, so $O \cup C$ gives a typical example of such composition. Let P^{kl} denote ADM momentum on O and $\Pi_{AB} := -\frac{1}{2}\sin\theta\partial_r(r\gamma_{AB} - r\overset{\circ}{\gamma}_{AB})$ be its equivalent on C. The conformally rescaled field $\psi^{AB} := r\gamma^{AB} - r\overset{\circ}{\gamma}{}^{AB}$ on C is a counterpart of riemannian metric[c] g_{kl} on O.

[c]This way the phase space on a surface $O \cup C$ consists of $\left[(P^{kl}, g_{kl}); (\Pi_{AB}, \psi^{AB})\right]$.

In the case of Einstein gravity the variation of the Hilbert Lagrangian $L = \frac{1}{16\pi}\sqrt{|g|}\, R$ (see [2] and [5]) leads to the following formula analogous to (2):

$$16\pi\delta m_{TB} = \int_O \dot{g}_{kl}\delta P^{kl} - \dot{P}^{kl}\delta g_{kl} + \int_C \dot{\psi}^{AB}\delta\Pi_{AB} - \dot{\Pi}_{AB}\delta\psi^{AB} + \frac{1}{2}\int_{S_u} \sin\theta\dot{\psi}_{AB}\delta\psi^{AB} \quad (4)$$

where the integral at null infinity $m_{TB} := \frac{1}{8\pi}\int_{S_u} r - V = \frac{1}{4\pi}\int_{S_u} M\sin\theta$ defines the energy in the radiating regime ($V = r - 2M + O(r^{-1})$)[d].

Equation (4) is the variational formula on a truncated cone $O \cup C$, which is space-like inside and light-like near Scri. One can also take a space-like hyperboloidal hypersurface Σ_u, which approaches $\mathscr{I}^+$ in an appropriate way by moving the sphere $S(u, r_0) = O \cap C$ to the null infinity along cone C ($O \subset \Sigma_u$). The above observations confirm the fact that Trautman-Bondi mass is not sensitive on the particular choice of the internal shape of the hypersurface but depends only on its boundary, which is a section of $\mathscr{I}^+$. Similarly as for the energy of the scalar field, one can denote the non-conservation law for the gravitational mass as follows $16\pi\partial_0 m_{TB} = \frac{1}{2}\int_{S_u}\sin\theta\dot{\psi}_{AB}\dot{\psi}^{AB}\ \left(= -\frac{1}{2}\int_{S_u}\sin\theta\, \overset{\circ}{\chi}_{AB,u}\, \overset{\circ}{\chi}{}^{AB}{}_{,u}\right)$ where the last form in the brackets becomes clear when we apply the asymptotics presented in [3]. In particular, $\psi_{AB}|_{\mathscr{I}^+} = \overset{\circ}{\chi}_{AB}$ and $\psi^{AB}|_{\mathscr{I}^+} = -\overset{\circ}{\chi}{}^{AB}$. This formula expresses the main result of the classical paper [1] and is valid in this form for much wider asymptotics than considered in the original papers. For angular momentum we obtain[e]

$$16\pi\partial_0 J_z = \frac{1}{2}\int_{S_u}\sin\theta\dot{\psi}_{AB}\partial_\varphi\psi^{AB}$$

where now

$$J_z := \int_O P^{kl}\partial_\varphi g_{kl} + \int_C \Pi_{AB}\partial_\varphi\psi^{AB}\,.$$

On the other hand one can easily check (see [3] p.715) that J_z is given as a boundary integral for the superpotential proposed by Komar.

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[d] The asymptotic behaviour of the full metric $g_{\mu\nu}$ in the form (3) is given in [3] and [4].

[e] The non-conservation laws for the energy and angular momentum can be derived from variational formula (4) simply by putting $\delta = \partial_0$ and $\delta = \partial_\varphi$ respectively.