THE GAUGE PRINCIPLE

IN THE 2D σ-MODEL

1. GENERAL STRUCTURES

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1. I.R. & R.R.S., Adv. Theor. Math. Phys. **13** (2009), 1137-1219.
2. K.G., R.R.S. & K.W., Commun. Math. Phys. **284** (2008), 1-49.
3. K.G., R.R.S. & K.W., Adv. Theor. Math. Phys. **15** (2011), in press.
4. R.R.S., Hamburger Beiträge zur Mathematik Nr. 360 (2011) [1101.1126 [hep-th]].
5. K.G., R.R.S. & K.W., Commun. Math. Phys. **302** (2010), 513-580.
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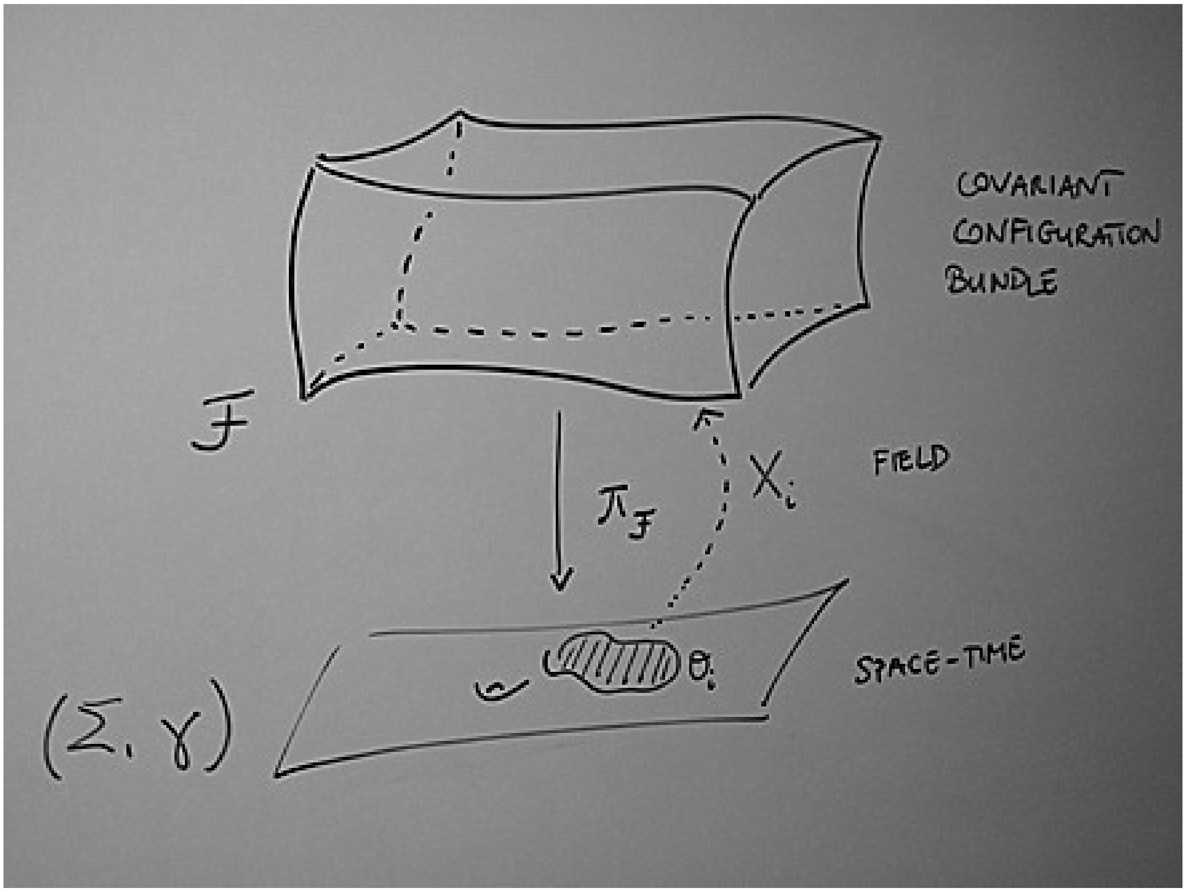
7. K.G., R.R.S. & K.W., "Gauging symmetries of 2d sigma-models on world-sheets with defects", in writing.
8. R.R.S., "Defects, dualities and the geometry of strings via gerbes, II. Generalised geometries with a twist", in writing.

I Introduction

Setting: FIELD THEORY



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ACTION FUNCTIONAL $S[X]$: $\delta S[X_{\rm cl.}] = 0$ (LAP)

RIGID SYMMETRIES:

$\mathscr{F}\ell : {\rm G} \times \mathscr{F} \to \mathscr{F} : (g, \sigma, X_i(\sigma)) \mapsto (\sigma, g.X_i(\sigma))$, ${\rm G} \subset \mathscr{D}iff(\mathscr{F})$,

i.e. $\mathscr{F}$ is a G-space, and

$$\delta S[g.X_{\rm cl.}] = 0 \,.$$

Idea: RIGID $\xrightarrow{\text{GAUGING}}$ LOCAL : ${\rm G} \ni g \xrightarrow{\sigma-\text{dep.}} g(\cdot) \in {\rm G}_\Sigma$

Problem: OBSTRUCTIONS $\equiv$ GAUGE ANOMALIES

- LOCAL/INFINITESIMAL : non-invariance of S under infinitesimal gauge transformations ($\stackrel{\int}{\to}$ homotopic to id)
- GLOBAL/TOPOLOGICAL : non-invariance of

FEYNMAN AMPLITUDES : $\mathcal{A}[X] = \mathrm{e}^{-S[X]}$

under large gauge transformations (non-homotopic to id)

Remark: The latter lead to destructive interferences of homotopy sectors within gauge orbits of states when gauge fields rendered dynamical.

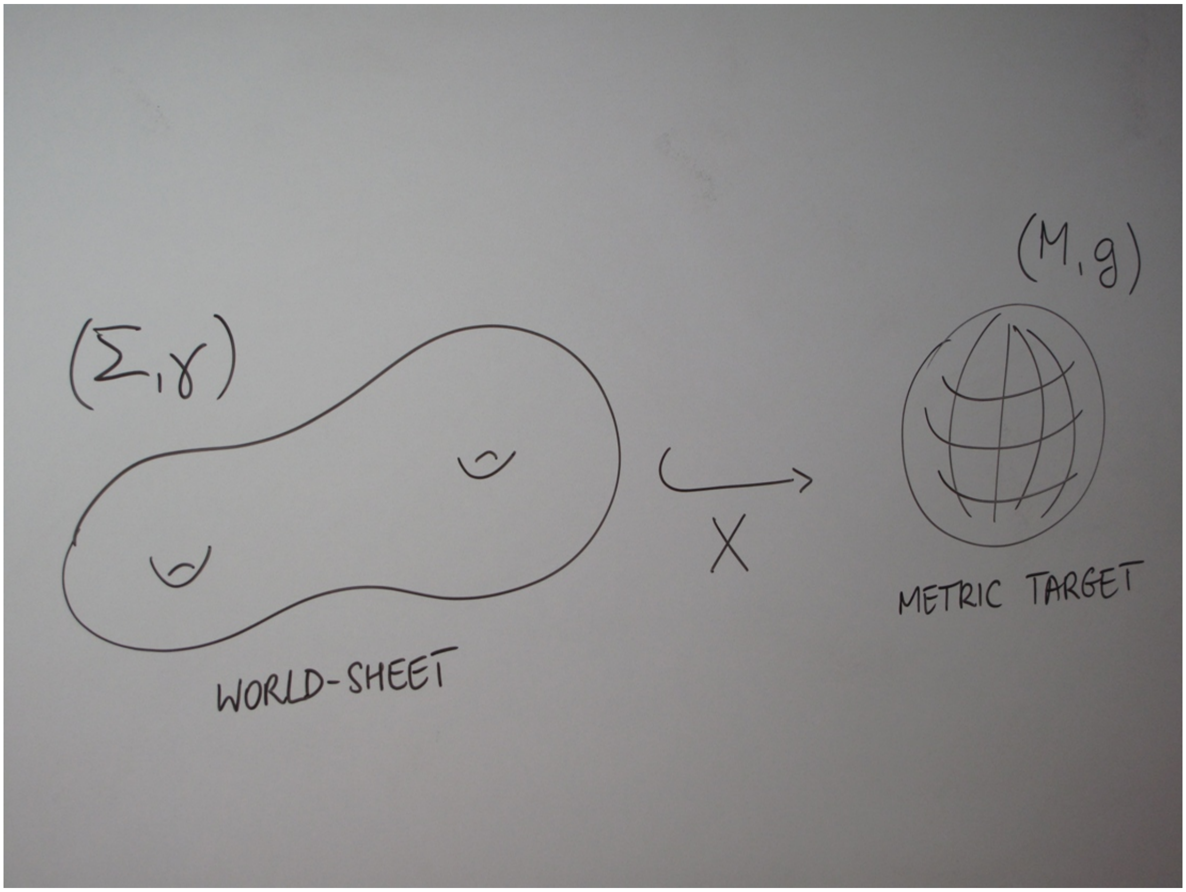
Ramifications:

- powerful selection rules for model building;
- effective field theories with $\mathscr{F}//\mathrm{G}$;
- exhaustive construction method for RCFT's;
- natural incorporation of BCFT's and CFT's over NON-ORIENTABLE WORLD-SHEETS (key rôle in susy extensions);
- T-DUALITY/MIRROR SYMMETRY.

II Requisites

(IIσ) Rudiments of the field theory

MONO-PHASE σ-MODEL:



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$$S_{\sigma,\mathrm{met.}}[X;\gamma] := -\tfrac{1}{2}\int_\Sigma \mathrm{g}(\mathsf{d}X \overset{\wedge}{,} \star_\gamma \mathsf{d}X) \quad \text{(to start with)}$$

Problem: WEYL ANOMALY $\sim$ $R_{\mu\nu}(\nabla_{\mathrm{L-C}}(\mathrm{g}))$

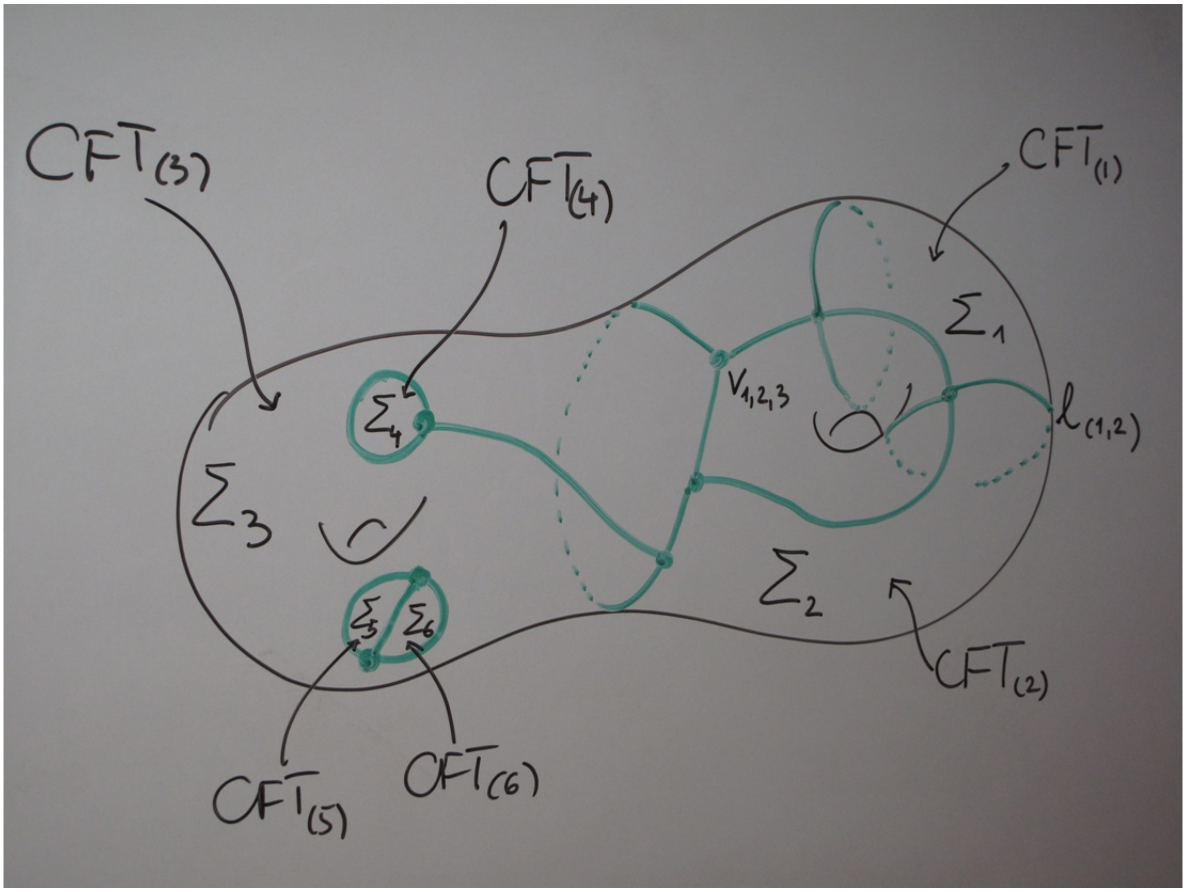
Solution: $\mathrm{H} \in Z^3(M)$: $R_{\mu\nu}(\nabla_{\mathrm{L-C}}(\mathrm{g})) = \tfrac{1}{4}\,\mathrm{H}_{\mu\alpha\beta}\,\mathrm{H}_{\nu\gamma\delta}\,(\mathrm{g}^{-1})^{\alpha\gamma}\,(\mathrm{g}^{-1})^{\beta\delta}$,

$S_\sigma[X;\gamma] := S_{\sigma,\mathrm{met.}}[X;\gamma] + S_{\sigma,\mathrm{WZ}}[X]$, $\quad S_{\sigma,\mathrm{WZ}}[X] := \mathsf{i}\int_\Sigma X^{*}\text{"}\mathsf{d}^{-1}\mathrm{H}\text{"}$.

Aside: WZ term consistent with STh

Problem: It may happen that $[\mathrm{H}]_{\mathrm{dR}} \neq 0$

In general, we have MULTI-PHASE σ-model



- patches $\Sigma_i \to (M_i, \mathrm{g}_i, \ldots)$, combine into $\mathcal{M} := \bigsqcup_i (M_i, \mathrm{g}_i, \ldots)$;

- DEFECT LINES $l_{i,j} \to (Q_{i,j}, \ldots)$, combine into $\mathcal{B} := \bigsqcup_{(i,j)} (Q_{i,j}, \ldots)$, w/ $\iota_1, \iota_2 : Q \to M$;

- DEFECT JUNCTIONS $v_{i_1,i_2,\ldots,i_n} \to (T_{i_1,i_2,\ldots,i_n}, \ldots)$, combine into $\mathcal{J} := \bigsqcup_{n\geq 3} \bigsqcup_{(i_1,i_2,\ldots,i_n)} (T_{i_1,i_2,\ldots,i_n}, \ldots)$, w/ $\pi_n^{k,k+1} : T_n \to Q$.

Relevance of defects:

- appear naturally in the orbifold σ-model (twisted sector);

- describe the most general CFT, incl. BCFT;

- capture symmetries/dualities of the σ-model.

<u>N.B.</u> The missing bits $(\ldots)$ come from 2-CATEGORY $\mathfrak{BGrb}^\nabla(\mathscr{F})$.

(IIg) Rudiments of gerbe theory

0. BUNDLE GERBE WITH CONNECTIVE STRUCTURE:

$$
\mathcal{G} \quad : \quad
\begin{array}{ccc}
(L,\nabla_L,\mu_L) & & \\
\downarrow{\scriptstyle \pi_L} & & \\
\mathsf{Y}^{[2]}\mathscr{F} & \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} & (\mathsf{Y}\mathscr{F}, B) \\
 & & \downarrow{\scriptstyle \pi_{\mathsf{Y}\mathscr{F}}} \\
 & & (\mathscr{F}, \mathrm{H})
\end{array}
$$

- CURVATURE $\quad \mathrm{curv}(\mathcal{G}) = \mathrm{H} \in Z^3(\mathscr{F}, 2\pi\mathbb{Z})$;
- CURVING $\quad \mathrm{B} \in \Omega^2(\mathsf{Y}\mathscr{F}) \ : \ \pi_{\mathsf{Y}\mathscr{F}}^*\mathrm{H} = \mathsf{d}B$;
- CONNECTION $\quad \nabla_L$ on L, with $\mathrm{curv}(\nabla_L) = B_{[2]} - B_{[1]}$;
- GROUPOID STRUCTURE $\quad \mu_L \,:\, L_{[1,2]} \otimes L_{[2,3]} \xrightarrow{\cong} L_{[1,3]}$.

Upshot: $\qquad \mathscr{A}_{\mathrm{WZ}}[X] \equiv \mathrm{Hol}_{\mathcal{G}}(X) = [X^*\mathcal{G}] \in \check{H}^2(\Sigma, \mathrm{U}(1)) \cong \mathrm{U}(1)$

1. GERBE 1-ISOMORPHISM: $\qquad \Phi \,:\, \mathcal{G}_1 \xrightarrow{\cong} \mathcal{G}_2$,

$$
\begin{array}{ccccc}
 & & (E,\nabla_E,\alpha) & & \\
 & & \downarrow{\scriptstyle \pi_E} & & \\
 & & \mathsf{Y}\mathsf{Y}_{1,2}\mathscr{F} & & \\
 & & \downarrow{\scriptstyle \pi_{\mathsf{Y}\mathsf{Y}_{1,2}\mathscr{F}}} & & \\
 & & \mathsf{Y}_1\mathscr{F} \times_{\mathscr{F}} \mathsf{Y}_2\mathscr{F} =: \mathsf{Y}_{1,2}\mathscr{F} & & \\
 & {\scriptstyle \mathrm{pr}_1}\swarrow & & \searrow{\scriptstyle \mathrm{pr}_2} & \\
\mathsf{Y}_1\mathscr{F} & & & & \mathsf{Y}_2\mathscr{F} \\
 & {\scriptstyle \pi_{\mathsf{Y}_1\mathscr{F}}}\searrow & & \swarrow{\scriptstyle \pi_{\mathsf{Y}_2\mathscr{F}}} & \\
 & & \mathscr{F} & &
\end{array}
$$

- CONNECTION $\quad \nabla_E$ on E, with $\mathrm{curv}(\nabla_E) = \pi_2^*B_2 - \pi_1^*B_1$;
- COHERENT ISO $\quad \alpha \,:\, E_{[1,2]} \otimes L_{2\,[2,4]} \xrightarrow{\cong} L_{1,[1,3]} \otimes E_{[3,4]}$

2. <u>GERBE 2-ISOMORPHISM:</u> $\varphi \doteq \mathcal{G}_1 \underset{\Phi_2}{\overset{\Phi_1}{\Downarrow\varphi}} \mathcal{G}_2$,

i.e. COHERENT ISO $\phi : E_{1\,[1]} \xrightarrow{\cong} E_{2\,[2]}$ over

$$\begin{array}{ccccc} & & \mathsf{YY}^{1,2}\mathsf{Y}_{1,2}\mathscr{F} & & \\ & & \downarrow{\scriptstyle \pi_{\mathsf{YY}^{1,2}\mathsf{Y}_{1,2}\mathscr{F}}} & & \\ & & \mathsf{Y}^1\mathsf{Y}_{1,2}\mathscr{F} \times_{\mathsf{Y}_{1,2}\mathscr{F}} \mathsf{Y}^2\mathsf{Y}_{1,2}\mathscr{F} =: \mathsf{Y}^{1,2}\mathsf{Y}_{1,2}\mathscr{F} & & \\ & \swarrow{\scriptstyle \mathrm{pr}_1} & & {\scriptstyle \mathrm{pr}_2}\searrow & \\ \mathsf{Y}^1\mathsf{Y}_{1,2}\mathscr{F} & & & & \mathsf{Y}^2\mathsf{Y}_{1,2}\mathscr{F} \\ & {\scriptstyle \pi_{\mathsf{Y}^1\mathsf{Y}_{1,2}\mathscr{F}}}\searrow & & \swarrow{\scriptstyle \pi_{\mathsf{Y}^2\mathsf{Y}_{1,2}\mathscr{F}}} & \\ & & \mathsf{Y}_{1,2}\mathscr{F} & & \end{array}$$

These compose STRICT MONOIDAL 2-CATEGORY $\mathfrak{BGrb}^\nabla(\mathscr{F})$

- 0-cells: gerbes;
- 1-cells: gerbe 1-isos (incl. canonical $\mathrm{Id}_\mathcal{G} : \mathcal{G} \xrightarrow{\cong} \mathcal{G}$), with assoc. HORIZONTAL COMPOSITION

$$\Phi_{2,3} \circ \Phi_{1,2} \ : \ \mathcal{G}_1 \xrightarrow{\Phi_{1,2}} \mathcal{G}_2 \xrightarrow{\Phi_{2,3}} \mathcal{G}_3$$

- $\mathrm{Hom}_{\mathfrak{BGrb}^\nabla(\mathscr{F})}(\mathcal{G}_1, \mathcal{G}_2)$ is a CATEGORY with morphisms
- 2-cells: gerbe 2-isos (incl. canonical $\mathrm{id}_{\mathrm{Id}_\mathcal{G}} : \mathrm{Id}_\mathcal{G} \overset{\cong}{\Longrightarrow} \mathrm{Id}_\mathcal{G}$), with assoc. COMPOSITIONs:

$$\varphi_2 \circ \varphi_1 \doteq \mathcal{G}_1 \underset{\Phi'_{1,2}}{\overset{\Phi_{1,2}}{\Downarrow\varphi_1}} \mathcal{G}_2 \underset{\Phi'_{1,2}}{\overset{\Phi_{2,3}}{\Downarrow\varphi_2}} \mathcal{G}_3 \qquad \text{(HORIZONTAL)}\,;$$

$$\varphi_2 \bullet \varphi_1 \doteq \mathcal{G}_1 \xrightarrow[\Phi''_{1,2}]{\Phi_{1,2}\ \Downarrow\varphi_1\ \Phi'_{1,2}\ \Downarrow\varphi_2} \mathcal{G}_2 \qquad \text{(VERTICAL)}\,.$$

subject to INTERCHANGE LAW.

Xtras:

- there exist distinguished TRIVIAL GERBEs $\quad\equiv\quad$ 2-forms:

$$I_\omega := (\mathscr{F}, \omega, \mathscr{F} \times \mathbb{C}, m)\,, \qquad\qquad m\left((x,z) \otimes (x,z')\right) := (x, z \cdot z')$$

- smooth maps $f : \mathscr{F}_1 \to \mathscr{F}_2$ give rise to PULLBACK (2-)FUNCTORS $f^* : \mathfrak{BGrb}^\nabla(\mathscr{F}_2) \to \mathfrak{BGrb}^\nabla(\mathscr{F}_1)$.

- neat description in terms of Deligne hypercohomology $\mathbb{H}^2\left(\mathscr{F}, \mathcal{D}(2)^\bullet_{\mathscr{F}}\right)$, i.e. cohomology of Čech-extended DELIGNE COMPLEX

$$\mathcal{D}(2)^\bullet \;:\; 0 \to \underline{\mathrm{U}(1)}_{\mathscr{F}} \xrightarrow{\frac{1}{\mathsf{i}}\mathsf{d}\log} \underline{\Omega}^1(\mathscr{F}) \xrightarrow{\mathsf{d}} \underline{\Omega}^2(\mathscr{F})$$

- there exists TRANSGRESSION FUNCTOR

$$\tau \;:\; \mathbb{H}^2\left(\mathscr{F}, \mathcal{D}(2)^\bullet_{\mathscr{F}}\right) \to \mathbb{H}^1\left(\mathsf{L}\mathscr{F}, \mathcal{D}(1)^\bullet_{\mathsf{L}\mathscr{F}}\right)$$

- we have

Proposition 1. *The following holds true in* $\mathfrak{BGrb}^\nabla(\mathscr{F})$*:*

(1) *Given* $\mathcal{G}_\alpha,\ \alpha = 1,2$ *with* $\mathrm{curv}(\mathcal{G}_1) = \mathrm{curv}(\mathcal{G}_2)$*, there exist*

$$\mathcal{G}_2 \xrightarrow{\cong} \mathcal{D} \otimes \mathcal{G}_1\,, \qquad\qquad \mathrm{curv}(\mathcal{D}) = 0\,,$$

and $\mathcal{G}_1 \cong \mathcal{G}_2$ *iff* $\mathcal{D} \cong I_0$.

(2) *Given* $\Psi_\alpha : \mathcal{G} \xrightarrow{\cong} \mathcal{H},\ \alpha = 1,2,$ *there exist*

$$\Psi_2 \xRightarrow{\cong} \mathrm{Bun}^{-1}(D) \otimes \Psi_1\,, \qquad\qquad \mathrm{curv}(D) = 0\,,$$

where $\mathrm{Bun} : \mathfrak{Bun}_0^\nabla(\mathscr{F}) \xrightarrow{\equiv} \mathfrak{End}(I_0)$ *is the canonical equivalence of categories, and* $\Phi_1 \cong \Phi_2$ *iff* $D \cong J_0$.

(3) *Given* $\psi_\alpha : \Psi \xRightarrow{\cong} \Xi,\ \alpha = 1,2,$ *there exists*

$$\psi_2 = d \otimes \psi_1\,, \qquad\qquad d \in C^\infty\left(\pi_0(\mathscr{F}), \mathrm{U}(1)\right)\,.$$

Definition 2. STRING BACKGROUND $\mathfrak{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J})$:

(1) **TARGET** $\mathcal{M} = (M, \mathrm{g}, \mathcal{G})$ with smooth metric **TARGET SPACE** (M, g), and bundle gerbe $\mathcal{G}$;

(2) $\mathcal{G}$**-BI-BRANE** $\mathcal{B} = \big(Q, \iota_1, \iota_2, \omega, \Phi\big)$ with smooth $\mathcal{G}$**-BI-BRANE WORLD-VOLUME** Q, $\mathcal{G}$**-BI-BRANE CURVATURE** $\omega \in \Omega^2(Q)$, smooth maps $\iota_1, \iota_2 : Q \to M$, and

$$\Phi \;:\; \iota_1^*\mathcal{G} \xrightarrow{\cong} \iota_2^*\mathcal{G} \otimes I_\omega\,;$$

(3) collection $\mathcal{J} = (\mathcal{J}_3, \mathcal{J}_4, \mathcal{J}_5, \ldots)$ of n**-VALENT** $(\mathcal{G}, \mathcal{B})$**-INTER-BI-BRANES**

$$\mathcal{J}_n = \big(T_n; \varepsilon_n^{1,2}, \varepsilon_n^{2,3}, \ldots, \varepsilon_n^{n-1,n}, \varepsilon_n^{n,1}; \pi_n^{1,2}, \pi_n^{2,3}, \ldots, \pi_n^{n-1,n}, \pi_n^{n,1}; \varphi_n\big)$$

consisting of:

(a) smooth $(\mathcal{G}, \mathcal{B})$**-IBB WORLD-VOLUMES** T_n;

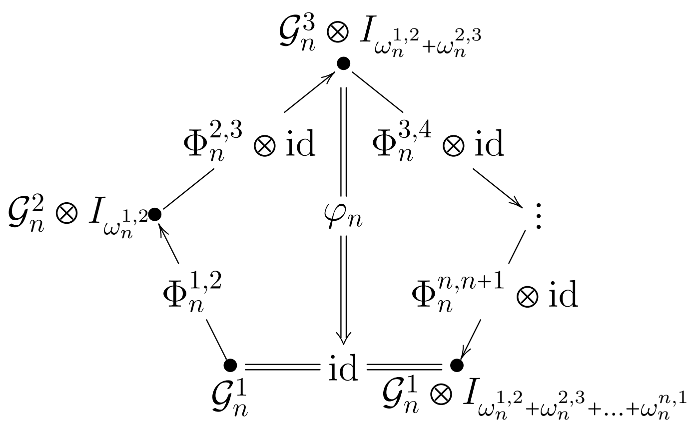
(b) **ORIENTATION MAPS** $\varepsilon_n^{k,k+1} : T_n \to \{-1, 1\}$;

(c) smooth maps $\pi_n^{k,k+1} : T_n \to Q$ subject to constraints

$$\iota_2^{\varepsilon_n^{k-1,k}} \circ \pi_n^{k-1,k} = \iota_1^{\varepsilon_n^{k,k+1}} \circ \pi_n^{k,k+1} =: \pi_n^k\,, \qquad k \in \mathbb{Z}/n\mathbb{Z}\,,$$

where $\iota_1^{+1} := \iota_1$, $\iota_2^{+1} := \iota_2$, $\iota_1^{-1} := \iota_2$ and $\iota_2^{-1} := \iota_1$;

(d) for $\Phi_n^{k,k+1} = (\pi_n^{k,k+1})^*\Phi^{\varepsilon_n^{k,k+1}}$, $\mathcal{G}_n^k = (\pi_n^k)^*\mathcal{G}$, and $\omega_n^{k,k+1} = \varepsilon_n^{k,k+1}\,(\pi_n^{k,k+1})^*\omega$,



$\mathscr{F} := M \sqcup Q \sqcup T$ is termed **TARGET SPACE of** $\mathfrak{B}$.

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Upshot: The many uses of $\mathfrak{B}$:

- determines FEYNMAN AMPLITUDE

$$\mathscr{A}[(X\,|\,\Gamma);\gamma] = \exp\left(-\frac{1}{2}\int_\Sigma \mathrm{g}\,(\mathsf{d}X \overset{\wedge}{,} \star_\gamma \mathsf{d}X)\right)\cdot \mathrm{Hol}_{\mathcal{G},\Phi,\varphi_n}(X\,|\,\Gamma)$$

as 2-DECORATED HOLONOMY for NETWORK-FIELD CONFIGURATIONS $(X\,|\,\Gamma)$.

- through transgression, induces PRE-QUANTUM BUNDLE

$$\mathbb{C} \hookrightarrow \mathcal{L}_{\mathfrak{B}} \to \mathsf{P}_\sigma^{(N)}\,, \qquad [\mathrm{curv}(\nabla_{\mathcal{L}_{\mathfrak{B}}})]_{\mathrm{dR}} = [\Omega_\sigma^{(N)}]_{\mathrm{dR}}\,,$$

for (pre-)SYMPLECTIC FORM $\Omega_\sigma^{(N)}$ on (N-TWISTED) PHASE SPACE $\mathsf{P}_\sigma^{(N)}$ (derived in GKST formalism).

III Rigid symmetries of the multi-phase σ-model

(IIIi) The infinitesimal picture

Consider ${}^{\mathscr{F}}\mathscr{K} := ({}^{M}\mathscr{K}\,, {}^{Q}\mathscr{K}\,, {}^{T_n}\mathscr{K}\,) \in \mathfrak{X}(\mathscr{F})$ subject to ALIGNMENT

$$\iota_{\alpha\,*}{}^{Q}\mathscr{K} = {}^{M}\mathscr{K}\big|_{\iota_\alpha(Q)}\,, \qquad \pi^{k,k+1}_{n\,*}{}^{T_n}\mathscr{K} = {}^{Q}\mathscr{K}\big|_{\pi^{k,k+1}_n(T_n)}\,.$$

Under their (local) flows $\xi_t : \mathscr{F} \to \mathscr{F}$,

$$\mathscr{A}[(X\,|\,\Gamma);\gamma]^{-1}\,\frac{\mathsf{d}}{\mathsf{d}t}\Big|_{t=0}\mathscr{A}[(\xi_t\circ X\,|\,\Gamma);\gamma]$$

$$= -\frac{1}{2}\int_\Sigma \big(\mathscr{L}_{{}^{M}\mathscr{K}}\mathrm{g}\big)\,(\mathsf{d}X \overset{\wedge}{,}\, \star_\gamma\,\mathsf{d}X) + \mathsf{i}\int_\Sigma X^*\,({}^{M}\mathscr{K}\lrcorner\,\mathrm{H}) + \mathsf{i}\int_\Gamma (X|_\Gamma)^*\,({}^{Q}\mathscr{K}\lrcorner\,\omega)\,.$$

Proposition 3. ${}^{\mathscr{F}}\mathscr{K}$ *engenders RIGID SYMMETRY iff*

$$\mathscr{L}_{{}^{M}\mathscr{K}}\mathrm{g} = 0\,, \qquad {}^{M}\mathscr{K}\lrcorner\,\mathrm{H} = -\mathsf{d}\kappa\,, \qquad \kappa \in \Omega^1(m)\,,$$

$${}^{Q}\mathscr{K}\lrcorner\,\omega + \Delta_Q\kappa = -\mathsf{d}k\,, \qquad k \in \Omega^0(Q) \qquad \Delta_{T_n}k = 0\,,$$

where $\Delta_Q := \iota_2^* - \iota_1^*$ *and* $\Delta_{T_n} := \sum_{k=1}^n \varepsilon_n^{k,k+1}\left(\pi_n^{k,k+1}\right)^*$.

The above combine into σ-SYMMETRIC SECTION

$$\mathfrak{K} := ({}^{M}\mathscr{K}\oplus\kappa, {}^{Q}\mathscr{K}\oplus k, {}^{T_n}\mathscr{K}\,) \in \Gamma_\sigma(\mathsf{E}\mathscr{F})$$

of GENERALISED TANGENT BUNDLES

$$\mathsf{E}\mathscr{F} := (\mathsf{T}M\oplus\mathsf{T}^*M)\sqcup(\mathsf{T}Q\oplus(Q\times\mathbb{R}))\sqcup\mathsf{T}T \to \mathscr{F}\,,$$

on which there exists $(\mathrm{H},\omega;\Delta_Q)$-TW. BRACKET STRUCTURE

$$\left(\mathsf{E}\mathscr{F}, [\![\,\cdot\,,\,\cdot\,]\!]^{(\mathrm{H},\omega;\Delta_Q)}, (\,\cdot\,,\,\cdot\,)_{\lrcorner}, \alpha_{\mathsf{T}\mathscr{F}}\right)\,,$$

generalising $(\mathsf{T}\mathscr{F},[\cdot,\cdot])$ and CLOSING ON $\Gamma_\sigma(\mathsf{E}\mathscr{F})$.

$(\mathrm{H},\omega;\Delta_Q)$-TW. BRACKET (of $\mathfrak{V}_i = ({}^{M}\mathscr{V}_i \oplus \upsilon_i, {}^{Q}\mathscr{V}_i \oplus f_i, {}^{T_n}\mathscr{V}_i)$)

$$\llbracket \mathfrak{V}_1 , \mathfrak{V}_2 \rrbracket^{(\mathrm{H},\omega;\Delta_Q)}|_M = [{}^{M}\mathscr{V}_1, {}^{M}\mathscr{V}_2] \oplus \Big(\mathscr{L}_{\mathscr{V}_1}\upsilon_2 - \mathscr{L}_{\mathscr{V}_2}\upsilon_1 - \tfrac{1}{2}\,\mathsf{d}({}^{M}\mathscr{V}_1 \lrcorner \upsilon_2 - {}^{M}\mathscr{V}_2 \lrcorner \upsilon_1) + \imath_{\mathscr{V}_1}\imath_{\mathscr{V}_2}\mathrm{H} \Big) ,$$

$$\llbracket \mathfrak{V}_1 , \mathfrak{V}_2 \rrbracket^{(\mathrm{H},\omega;\Delta_Q)}|_Q = [{}^{Q}\mathscr{V}_1, {}^{Q}\mathscr{V}_2] \oplus \Big(\mathscr{L}_{\mathscr{V}_1} f_2 - \mathscr{L}_{\mathscr{V}_2} f_1 + {}^{Q}\mathscr{V}_1 \lrcorner {}^{Q}\mathscr{V}_2 \lrcorner \omega + \tfrac{1}{2}\,({}^{Q}\mathscr{V}_1 \lrcorner \Delta_Q\upsilon_2 - {}^{Q}\mathscr{V}_2 \lrcorner \Delta_Q\upsilon_1) \Big) ,$$

$$\llbracket \mathfrak{V}_1 , \mathfrak{V}_2 \rrbracket^{(\mathrm{H},\omega;\Delta_Q)}|_{T_n} = [{}^{T_n}\mathscr{V}_1, {}^{T_n}\mathscr{V}_2] .$$

Physical interpretation: On $\mathsf{P}_\sigma^{(N)}$, (pre-)symplectic form reads

$$\Omega_\sigma^{(N)} = \mathrm{pr}^*_{\mathsf{T}^*C^\infty(\mathbb{S}^1_{(N)},M)} \left(\delta\theta_{\mathsf{T}^*C^\infty(\mathbb{S}^1_{(N)},M)} + \pi^*_{\mathsf{T}^*C^\infty(\mathbb{S}^1_{(N)},M)} \int_{\mathbb{S}^1_{(N)}} \mathrm{ev}^*\mathrm{H} \right) + \int_{k=1}^{N} \varepsilon_k \,\mathrm{pr}^*_{Q^{(k)}}\omega ,$$

and we find NOETHER MAP

$$\mathscr{N} \ : \ \Gamma_\sigma(\mathsf{E}\mathscr{F}) \to \Gamma_{\mathrm{H}}\left(\mathsf{E}^{(1,0)}\mathsf{P}_\sigma^{(N)}\right) \ : \ \mathfrak{K} \mapsto \mathscr{X}_{\mathfrak{K}} \oplus h_{\mathfrak{K}}$$

for can. lifts $\mathscr{X}_{\mathfrak{K}}$ of $\alpha_{\mathsf{T}\mathscr{F}}(\mathfrak{K})$ and NOETHER HAMILTONIANS

$$h_{\mathfrak{K}} \in C^\infty(\mathsf{P}_\sigma^{(N)},\mathbb{R}) \qquad : \qquad \mathscr{X}_{\mathfrak{K}} \lrcorner \,\Omega_\sigma^{(N)} = -\delta h_{\mathfrak{K}} .$$

The latter are in INVOLUTION with respect to $\Omega_\sigma^{(N)}$-TW. VINOGRADOV BRACKET

$$[\, \mathscr{X}_1 \oplus f_1 \, , \, \mathscr{X}_2 \oplus f_2 \,]_{\mathrm{V}}^{\Omega_\sigma^{(N)}} := [\mathscr{X}_1, \mathscr{X}_2] \oplus \left(\mathscr{X}_1(f_2) - \mathscr{X}_2(f_1) + \mathscr{X}_1 \lrcorner \mathscr{X}_1 \lrcorner \Omega_\sigma^{(N)} \right) .$$

Proposition 4.

$$\mathscr{N} \ : \ \left(\Gamma_\sigma(\mathsf{E}\mathscr{F}), \llbracket \cdot , \cdot \rrbracket^{(\mathrm{H},\omega;\Delta_Q)}\right) \to \left(\Gamma\left(\mathsf{E}^{(1,0)}\mathsf{P}_\sigma^{(N)}\right), [\,\cdot\, , \cdot\,]_{\mathrm{V}}^{\Omega_\sigma^{(N)}}\right)$$

is a homomorphism of Lie algebras over $\mathbb{R}$.

Comments: $(\mathrm{H},\omega;\Delta_Q)$-twisted bracket structure

- restricts to Courant algebroid H-twisted à la Ševera–Weinstein;
- admits a natural gerbe-theoretic interpretation.

(IIIg) The global picture

In what follows, we shall also ultimately deal with "integrated" rigid symmetries.

Definition 5. LIE GROUPOID $\mathrm{Gr} = (\mathrm{Ob}\,\mathrm{Gr}, \mathrm{Mor}\,\mathrm{Gr}, s, t, \mathrm{Id}, \mathrm{Inv}, \circ)$ consists of

- smooth OBJECT SET $\mathrm{Ob}\,\mathrm{Gr}$;
- smooth ARROW SET $\mathrm{Mor}\,\mathrm{Gr}$;
- smooth STRUCTURE MAPS:
  - SOURCE $s : \mathrm{Mor}\,\mathrm{Gr} \to \mathrm{Ob}\,\mathrm{Gr}$ (surj. subm.);
  - TARGET $t : \mathrm{Mor}\,\mathrm{Gr} \to \mathrm{Ob}\,\mathrm{Gr}$ (surj. subm.);
  - UNIT $\mathrm{Id} : \mathrm{Ob}\,\mathrm{Gr} \to \mathrm{Mor}\,\mathrm{Gr} : m \mapsto \mathrm{Id}_m$;
  - INVERSE $\mathrm{Inv} : \mathrm{Mor}\,\mathrm{Gr} \to \mathrm{Mor}\,\mathrm{Gr} : \overrightarrow{g} \mapsto \overrightarrow{g}^{-1} \equiv \mathrm{Inv}(\overrightarrow{g})$;
  - MULTIPLICATION $\circ : \mathrm{Mor}\,\mathrm{Gr}_s{\times}_t\,\mathrm{Mor}\,\mathrm{Gr} \to \mathrm{Mor}\,\mathrm{Gr} : (\overrightarrow{g}, \overrightarrow{h}) \mapsto \overrightarrow{g} \circ \overrightarrow{h}$;

subject to consistency constraints:

(i) $s(\overrightarrow{g} \circ \overrightarrow{h}) = s(\overrightarrow{h}),\ t(\overrightarrow{g} \circ \overrightarrow{h}) = t(\overrightarrow{g})$;

(ii) $(\overrightarrow{g} \circ \overrightarrow{h}) \circ \overrightarrow{k} = \overrightarrow{g} \circ (\overrightarrow{h} \circ \overrightarrow{k})$;

(iii) $\mathrm{Id}_{t(\overrightarrow{g})} \circ \overrightarrow{g} = \overrightarrow{g} = \overrightarrow{g} \circ \mathrm{Id}_{s(\overrightarrow{g})}$;

(iv) $s(\overrightarrow{g}^{-1}) = t(\overrightarrow{g}),\ t(\overrightarrow{g}^{-1}) = s(\overrightarrow{g}),\ \overrightarrow{g} \circ \overrightarrow{g}^{-1} = \mathrm{Id}_{t(\overrightarrow{g})},\ \overrightarrow{g}^{-1} \circ \overrightarrow{g} = \mathrm{Id}_{s(\overrightarrow{g})}$.

✓

N.B. A (Lie) groupoid is a small category, and a Lie group is a Lie groupoid with a singleton as object set.

Relation to the infinitesimal picture shall be established through

Definition 6. LIE ALGEBROID $\mathfrak{Gr} = \big(V, [\cdot,\cdot], \alpha_{\mathsf{T}\mathscr{F}}\big)$ over smooth BASE $\mathscr{F}$ consists of

- vector bundle $\pi_V : V \to \mathscr{F}$;
- Lie bracket $[\cdot,\cdot]$ on $\Gamma(V)$;
- ANCHOR (bundle map) $\alpha_{\mathsf{T}\mathscr{F}} : V \to \mathsf{T}\mathscr{F}$

with the following properties:

(i) induced map $\Gamma(\alpha_{\mathsf{T}\mathscr{F}}) : \Gamma(V) \to \Gamma(\mathsf{T}\mathscr{F})$ is a Lie-algebra homomorphism;

(ii) Leibniz identity (for all $X, Y \in \Gamma(V)$ and $f \in C^\infty(\mathscr{F}, \mathbb{R})$)

$$[X, f\,Y] = f\,[X,Y] + \Gamma(\alpha_{\mathsf{T}\mathscr{M}})(X)(f)\,Y\,.$$

✓

and

Definition 7. Let $\mathrm{Gr} = (\mathrm{Ob}\,\mathrm{Gr}, \mathrm{Mor}\,\mathrm{Gr}, s, t, \mathrm{Id}, \mathrm{Inv}, \circ)$ be a Lie groupoid,

$$R_{\vec{g}} \;:\; s^{-1}(\{t(\vec{g})\}) \to s^{-1}(\{s(\vec{g})\}) \;:\; \vec{h} \mapsto R_{\vec{g}}(\vec{h}) := \vec{h} \circ \vec{g}\,,$$

and $\mathfrak{X}^s_{\mathrm{inv}}(\mathrm{Mor}\,\mathrm{Gr}) = \{\ \mathscr{V} \in \Gamma\big(\ker \mathsf{d}s\big) \ \mid\ \mathsf{d}R_\cdot(\mathscr{V}) = \mathscr{V}\ \}$ space of right Gr-invariant vector fields on Mor Gr. TANGENT ALGEBROID of Gr is $\mathfrak{gr} = \big(\mathrm{Id}^*\ker \mathsf{d}s, [\,\cdot\,,\,\cdot\,], \alpha_{\mathsf{T}(\mathrm{Ob}\,\mathrm{Gr})}\big)$. Anchor $\alpha_{\mathsf{T}(\mathrm{Ob}\,\mathrm{Gr})}$ induces map $\mathsf{d}t \circ i$ between spaces of sections, defined in terms of canonical vector-space isomorphism

$$i \;:\; \Gamma(\mathrm{Id}^*\ker \mathsf{d}s) \xrightarrow{\cong} \mathfrak{X}^s_{\mathrm{inv}}(\mathrm{Mor}\,\mathrm{Gr})\,,$$

and Lie bracket is the unique bracket on $\Gamma(\mathrm{Id}^*\ker \mathsf{d}s)$ for which i is an isomorphism of Lie algebras.

✓

In the case of interest, we encounter ACTION GROUPOID

$$\mathrm{G}\ltimes\mathscr{F} := \left(\mathscr{F}, \mathrm{G}\times\mathscr{F}, \mathrm{pr}_2, {}^{\mathscr{F}}\ell, \mathrm{Id}, \mathrm{Inv}, \circ\right)$$

with structure maps

$$\mathrm{Id}_m := (e,m)\,, \qquad\qquad \mathrm{Inv}(g,m) := (g^{-1}, g.m)\ ,$$

$$(h, g.m)\circ(g,m) := (h\cdot g, m)\,.$$

Its tangent algebroid, termed ACTION ALGEBROID, is

$$\mathfrak{g}\ltimes\mathscr{F} := \left(\bigoplus_{A=1}^{\dim_{\mathbb{R}}\mathfrak{g}} C^\infty(\mathscr{F},\mathbb{R})\,\mathscr{R}_A, [\cdot,\cdot]_{\mathfrak{g}\ltimes\mathscr{F}}, \alpha_{\mathsf{T}\mathscr{F}}\right)\,,$$

with Lie bracket

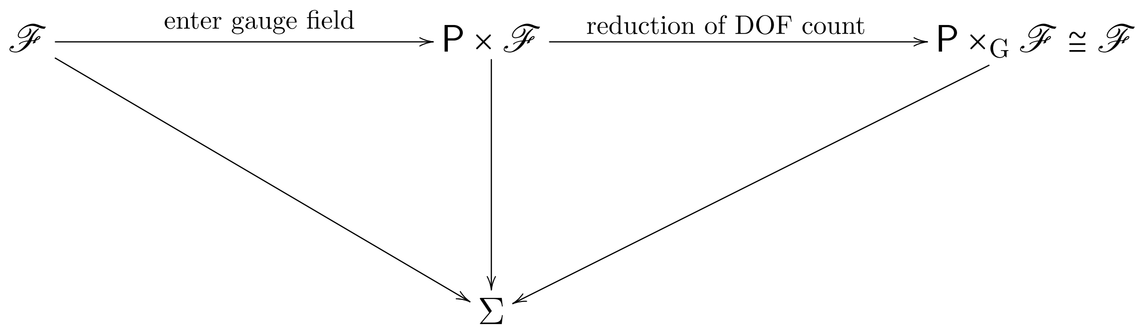
$$[\lambda^A\,\mathscr{R}_A, \mu^B\,\mathscr{R}_B]_{\mathfrak{g}\ltimes\mathscr{F}} := f_{ABC}\,\lambda^A\,\mu^B\,\mathscr{R}_C + \left(\mathscr{L}_{\lambda^A\,{}^{\mathscr{F}}\mathscr{K}_A}\mu^B - \mathscr{L}_{\mu^A\,{}^{\mathscr{F}}\mathscr{K}_A}\lambda^B\right)\mathscr{R}_B$$

and anchor

$$\alpha_{\mathsf{T}\mathscr{F}}(\mathscr{R}_A) := {}^{\mathscr{F}}\mathscr{K}_A\,.$$

A taste of the stuff to come…

Idea:



for $\mathsf{P} \to \Sigma$ *arbitrary* principal G-bundle with principal G-connection $\mathcal{A} \in \Omega^1(\mathsf{P}) \otimes \mathfrak{g}$, $\mathfrak{g} = \mathrm{Lie}\,\mathrm{G}$

Sketch of construction:

Step 1. Finding consistent coupling between $\mathfrak{B}$ and $\mathcal{A}$.

Problem: Minimal-coupling recipe fails in general.

Step 2. Lifting geometric G-action from $\mathscr{F}$ to $\mathfrak{B}$.

Problem: Obstructions to equivariantisation.

Step 3. Descending coupled string background from $\mathsf{P}\times\mathscr{F}$ to associated bundle $\mathsf{P}\times_{\mathrm{G}}\mathscr{F}$, and subsequently (whenever admissible) to coset $(\mathsf{P}\times_{\mathrm{G}}\mathscr{F})/\mathrm{G}$.

Problem: None, "miraculously"!

Guiding principle: The Principle of Categorial Descent.