Heavy scalar resonances in the alignment limit of the 2HDM

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- Motivation
- Brief introduction into the 2HDM
- Implications of the LHC Higgs signal:
  - The H_1VV coupling as in the SM - the alignment limit
  - CP-violation in the alignment limit requires the most general 2HDM
  - Decoupling v.s. non-decoupling of H_2, H_3 and $H^\pm$
  - "Heavy" Higgs boson $(H_2, H_3, H^\pm)$ properties in the alignment limit
- Testing the alignment limit of the 2HDM at the LHC
- Spontaneous CP violation
- Summary

- B.G., H. Haber, O. M. Ogreid and P. Osland, "Heavy Higgs boson decays in the alignment limit of the 2HDM", JHEP 1812 (2018) 056,
- B.G., O. M. Ogreid and P. Osland, "Spontaneous CP violation in the 2HDM: physical conditions and the alignment limit", Phys.Rev. D94 (2016) no.11, 115002,
- B.G., O. M. Ogreid and P. Osland, "CP-Violation in the ZZZ and ZW^+W^- vertices at e^+e^- colliders in Two-Higgs-Doublet Models", JHEP 1605 (2016) 025,
- B.G., O. M. Ogreid and P. Osland, "Measuring CP violation in Two-Higgs-Doublet models in light of the LHC Higgs data", JHEP 1411 (2014) 084,
- B.G., O. M. Ogreid and P. Osland, "Diagnosing CP properties of the 2HDM", JHEP **1401** (2014) 105.

Brief introduction into the 2HDM

Why 2HDM?

- Baryon asymmetry and the Sakharov conditions for baryogenesis
  - Baryon number non-conservation,
  - C- and CP-violation,
  - Thermal inequilibrium,

  Extra sources of CP-violation are required!

- Possibility of large (tree-level generated) FCNC, e.g. $t \to cH$ decays, interesting non-standard flavour physics

- 2HDM provides a framework for light new physics (light "heavy" Higgs bosons) that is easily tolerated by the Higgs boson discovery.
  see e.g.
  P.M. Ferreira, R. Santos, M. Sher, J. P. Silva, "Implications of the LHC two-photon signal for two-Higgs-doublet models", Phys.Rev. D85 (2012) 077703
  J. Bernon, J. Gunion, H. Haber, Y. Jiang, S. Kraml, "Scrutinizing the alignment limit in two-Higgs-doublet models: $m_h = 125$ GeV", Phys.Rev. D92 (2015) no.7, 075004

The 2HDM potential:

$$
\begin{aligned}
V(\phi_1,\phi_2) \quad = \quad & -\frac{1}{2}\left\{m_{11}^2\phi_1^\dagger\phi_1 + m_{22}^2\phi_2^\dagger\phi_2 + \left[m_{12}^2\phi_1^\dagger\phi_2 + \text{H.c.}\right]\right\} + \frac{1}{2}\lambda_1(\phi_1^\dagger\phi_1)^2 \\
& + \frac{1}{2}\lambda_2(\phi_2^\dagger\phi_2)^2 + \lambda_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) \\
& + \frac{1}{2}\left[\lambda_5(\phi_1^\dagger\phi_2)^2 + \text{H.c.}\right] + \left[\lambda_6(\phi_1^\dagger\phi_1) + \lambda_7(\phi_2^\dagger\phi_2)\right]\left[(\phi_1^\dagger\phi_2) + \text{H.c.}\right]
\end{aligned}
$$

Yukawa couplings:

$$
-\mathcal{L}_Y^{\text{quarks}} = \overline{Q_L^0}\left[\tilde{\Phi}_1\eta_1^{U,0} + \tilde{\Phi}_2\eta_2^{U,0}\right]U_R^0 + \overline{Q_L^0}\left[\Phi_1\left(\eta_1^{D,0}\right)^\dagger + \Phi_2\left(\eta_2^{D,0}\right)^\dagger\right]D_R^0 + \text{h.c.}
$$

with $\tilde{\Phi}_j = i\sigma_2\Phi_j^*$

$$
M_U^0 = \langle\tilde{\Phi}_1\rangle\eta_1^{U,0} + \langle\tilde{\Phi}_2\rangle\eta_2^{U,0} \qquad M_D^0 = \langle\Phi_1\rangle\eta_1^{D,0} + \langle\Phi_2\rangle\eta_2^{D,0}
$$

The type II model: $\mathbb{Z}_2$ softly broken (by $m_{12}^2 \neq 0$): $\Phi_1 \to -\Phi_1$ and $d_R \to -d_R \Rightarrow$ $\lambda_6 = \lambda_7 = 0$, $\eta_1^{U,0} = \eta_2^{D,0} = 0$. Here the most general 2HDM will be considered.

In an arbitrary basis, the vevs may be complex, and the Higgs-doublets can be written as

$$\Phi_j = e^{i\xi_j} \begin{pmatrix} \varphi_j^+ \\ (v_j + \eta_j + i\chi_j)/\sqrt{2} \end{pmatrix}, \quad j = 1, 2.$$

Here v_j are real numbers, so that $v_1^2 + v_2^2 = v^2$. The fields η_j and χ_j are real. The phase difference between the two vevs is defined as

$$\xi \equiv \xi_2 - \xi_1.$$

Next, let's define the Goldston bosons G_0 and $G^\pm$ by an orthogonal rotation

$$\begin{pmatrix} G_0 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} c_\beta & s_\beta \\ -s_\beta & c_\beta \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \qquad \begin{pmatrix} G^\pm \\ H^\pm \end{pmatrix} = \begin{pmatrix} c_\beta & s_\beta \\ -s_\beta & c_\beta \end{pmatrix} \begin{pmatrix} \varphi_1^\pm \\ \varphi_2^\pm \end{pmatrix}$$

where $s_\beta \equiv \sin\beta$ and $c_\beta \equiv \cos\beta$ for $\tan\beta \equiv v_2/v_1$. Then G_0 and $G^\pm$ become the massless Goldstone fields. $H^\pm$ are the charged scalars.

The model contains three neutral scalar mass-eigenstates, which are linear compositions of the η_i,

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = R \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix},$$

with the 3×3 orthogonal rotation matrix R satisfying

$$R \mathcal{M}^2 R^{\mathrm{T}} = \mathcal{M}^2_{\mathrm{diag}} = \mathrm{diag}(M_1^2, M_2^2, M_3^2),$$

and with $M_1 \leq M_2 \leq M_3$. A convenient parametrization of the rotation matrix R is

$$R = R_3 R_2 R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{pmatrix} \begin{pmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1\, c_2 & s_1\, c_2 & s_2 \\ -(c_1\, s_2\, s_3 + s_1\, c_3) & c_1\, c_3 - s_1\, s_2\, s_3 & c_2\, s_3 \\ -c_1\, s_2\, c_3 + s_1\, s_3 & -(c_1\, s_3 + s_1\, s_2\, c_3) & c_2\, c_3 \end{pmatrix}$$

Comments on parameters of the 2HDM

Invariants under $U(2)$ basis rotations:

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\Phi}_1 \\ \bar{\Phi}_2 \end{pmatrix} = e^{i\psi} \begin{pmatrix} \cos\theta & e^{-i\xi}\sin\theta \\ -e^{i\chi}\sin\theta & e^{i(\chi-\xi)}\cos\theta \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}.$$

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- M. Maniatis, A. von Manteuffel and O. Nachtmann, "CP violation in the general two-Higgs-doublet model: A Geometric view", Eur. Phys. J. C 57 (2008) 719 [arXiv:0707.3344 [hep-ph]].
- G. C. Branco, L. Lavoura and J. P. Silva, "CP Violation", Int. Ser. Monogr. Phys. 103, 1 (1999).

The potential contains 14 real parameters, our input parameters are

$$\mathcal{P}_{67} \equiv \{M_{H^\pm}^2, \mu^2, M_1^2, M_2^2, M_3^2, \mathrm{Im}\lambda_5, \mathrm{Re}\lambda_6, \mathrm{Re}\lambda_7, v_1, v_2, \xi, \alpha_1, \alpha_2, \alpha_3\},$$

where μ^2 is defined as $\mathrm{Re}\, m_{12}^2 \equiv \frac{2v_1v_2}{v^2}\mu^2$ and the extrema conditions are:

$$\begin{aligned}
m_{11}^2 &= v_1^2\lambda_1 + v_2^2(\lambda_3+\lambda_4) + \frac{v_2^2}{c_\xi}(\mathrm{Re}\,\lambda_5 c_\xi - \mathrm{Im}\,\lambda_5 s_\xi) \\
&\quad + \frac{v_1v_2}{c_\xi}\left[\mathrm{Re}\,\lambda_6(2+c_{2\xi}) - \mathrm{Im}\,\lambda_6 s_{2\xi}\right] + \frac{v_2}{v_1 c_\xi}(v_2^2\mathrm{Re}\,\lambda_7 - \mathrm{Re}\, m_{12}^2), \\
m_{22}^2 &= v_2^2\lambda_2 + v_1^2(\lambda_3+\lambda_4) + \frac{v_1^2}{c_\xi}(\mathrm{Re}\,\lambda_5 c_\xi - \mathrm{Im}\,\lambda_5 s_\xi) \\
&\quad + \frac{v_1v_2}{c_\xi}\left[\mathrm{Re}\,\lambda_7(2+c_{2\xi}) - \mathrm{Im}\,\lambda_7 s_{2\xi}\right] + \frac{v_1}{v_2 c_\xi}(v_1^2\mathrm{Re}\,\lambda_6 - \mathrm{Re}\, m_{12}^2), \\
\mathrm{Im}\, m_{12}^2 &= \frac{v_1v_2}{c_\xi}(\mathrm{Re}\,\lambda_5 s_{2\xi} + \mathrm{Im}\,\lambda_5 c_{2\xi}) + \frac{v_1^2}{c_\xi}(\mathrm{Re}\,\lambda_6 s_\xi + \mathrm{Im}\,\lambda_6 c_\xi) \\
&\quad + \frac{v_2^2}{c_\xi}(\mathrm{Re}\,\lambda_7 s_\xi + \mathrm{Im}\,\lambda_7 c_\xi) - \mathrm{Re}\, m_{12}^2 t_\xi,
\end{aligned}$$

with $c_x = \cos x$, $s_x = \sin x$, and $t_x = \tan x$.

In spite of the presence of 14 parameters present in

$$\mathcal{P}_{67} \equiv \{M_{H^\pm}^2, \mu^2, M_1^2, M_2^2, M_3^2, \mathrm{Im}\lambda_5, \mathrm{Re}\lambda_6, \mathrm{Re}\lambda_7, v_1, v_2, \xi, \alpha_1, \alpha_2, \alpha_3\},$$

observables depend on 11 weak-basis invariant parameters only, possible choice is

$$\mathcal{P} \equiv \{M_{H^\pm}^2, M_1^2, M_2^2, M_3^2, e_1, e_2, e_3, q_1, q_2, q_3, q\},$$

where

$$\begin{aligned} e_i \equiv v_1 R_{i1} + v_2 R_{i2} &\propto H_i V_\mu V^\mu, \\ q_i &\propto H_i H^+ H^-, \\ q &\propto H^+ H^+ H^- H^- \end{aligned}$$

Complex couplings (both scalar and gauge) contain the quantities

$$f_i \equiv v_1 R_{i2} - v_2 R_{i1} - i v R_{i3},$$

which are pseudo-invariants under basis transformation: $f_i \to e^{i\alpha} f_i$, which are not measurable, however f_i will always appear paired with f_j^*, and satisfy

$$f_i f_j^* = v^2 \delta_{ij} - e_i e_j + i v \epsilon_{ijk} e_k.$$

Gauge couplings:

$$\boxed{H_i Z_\mu Z_\nu : \quad \frac{ig^2}{2\cos^2\theta_{\mathrm{W}}} e_i \, g_{\mu\nu}, \qquad H_i W^+_\mu W^-_\nu : \quad \frac{ig^2}{2} e_i \, g_{\mu\nu}}$$

where

$$e_i \equiv v_1 R_{i1} + v_2 R_{i2}$$

In terms of the mixing angles

$$\begin{aligned}
e_1 &= \quad v \cos\alpha_2 \cos(\beta - \alpha_1) \\
e_2 &= \quad v[\cos\alpha_3 \sin(\beta - \alpha_1) - \sin\alpha_2 \sin\alpha_3 \cos(\beta - \alpha_1)] \\
e_3 &= -v[\sin\alpha_3 \sin(\beta - \alpha_1) + \sin\alpha_2 \cos\alpha_3 \cos(\beta - \alpha_1)]
\end{aligned}$$

Note that

$$e_1^2 + e_2^2 + e_3^2 = v^2$$

$$\boxed{(Z^\mu H_i H_j) : \quad \frac{g}{2v\cos\theta_{\mathrm{W}}} \epsilon_{ijk} e_k (p_i - p_j)^\mu,}$$

Scalar couplings:

$$\boxed{H_i H^- H^+ : \quad -i q_i}$$

where

$$q_i = \frac{2e_i}{v^2} M_{H^\pm}^2 - \frac{R_{i2}v_1 + R_{i1}v_2 - R_{i3} v t_\xi}{v_1 v_2 c_\xi} \mu^2 + \frac{g_i - R_{i3} v^3 t_\xi}{v^2 v_1 v_2} M_i^2 + \frac{R_{i3} v^3}{2 v_1 v_2 c_\xi^2} \mathrm{Im}\,\lambda_5$$
$$- \frac{v^2 \left(R_{i3} v t_\xi - R_{i2} v_1 + R_{i1} v_2\right)}{2 v_2^2 c_\xi} \mathrm{Re}\,\lambda_6 - \frac{v^2 \left(R_{i3} v t_\xi + R_{i2} v_1 - R_{i1} v_2\right)}{2 v_1^2 c_\xi} \mathrm{Re}\,\lambda_7$$

and $g_i \equiv v_1^3 R_{i2} + v_2^3 R_{i1}$.

$$\boxed{H^+ H^+ H^- H^- : \quad -4iq}$$

where

$$q = -\frac{1}{2v^2 v_1^2 v_2^2} \left(v_1^2 - v_2^2\right)^2 \mu^2 + \sum_{k=1}^{3} \frac{g_k^2}{2 v^4 v_1^2 v_2^2} M_k^2 + \frac{v^2 \left(v_1^2 - 3v_2^2\right)}{4 v_1 v_2^3} \mathrm{Re}\,\lambda_6 + \frac{v^2 \left(v_2^2 - 3v_1^2\right)}{4 v_2 v_1^3} \mathrm{Re}\,\lambda_7$$

Yukawa couplings:

$$-\mathcal{L}_Y^{\text{quarks}} = \overline{Q_L^0}\left[\tilde{\Phi}_1\eta_1^{U,0} + \tilde{\Phi}_2\eta_2^{U,0}\right]U_R^0 + \overline{Q_L^0}\left[\Phi_1\left(\eta_1^{D,0}\right)^\dagger + \Phi_2\left(\eta_2^{D,0}\right)^\dagger\right]D_R^0 + \text{h.c.}$$

with $\tilde{\Phi}_j = i\sigma_2\Phi_j^*$

$$M_U^0 = \langle\tilde{\Phi}_1\rangle\eta_1^{U,0} + \langle\tilde{\Phi}_2\rangle\eta_2^{U,0} \qquad M_D^0 = \langle\Phi_1\rangle\eta_1^{D,0} + \langle\Phi_2\rangle\eta_2^{D,0}$$

$$\Downarrow$$

FCNC

$$\Phi_j = \begin{pmatrix} \Phi_j^+ \\ \Phi_j^0 \end{pmatrix}$$

$$\begin{aligned} -\mathcal{L}_Y^{\text{quarks}} \quad = \quad & \overline{U}_L\left(\Phi_i^0\right)^*\eta_i^U U_R + \overline{D}_L\Phi_i^0\left(\eta_i^D\right)^\dagger D_R \\ & -\overline{D}_L K^\dagger\Phi_i^-\eta_i^U U_R + \overline{U}_L K\Phi_i^+\left(\eta_i^D\right)^\dagger D_R + \text{h.c.}, \end{aligned}$$

where K is the CKM matrix. Next, we decompose these η_i-matrices into a part κ proportional to the masses, and an orthogonal part

$$\eta_i^Q = \kappa^Q \hat{v}_i + \rho^Q \hat{w}_i$$

with $Q = U, D$. Here

$$\begin{aligned} \hat{v}_j &= \frac{v_j}{v} e^{i\xi_j} \\ \hat{w}_1 &= -\frac{v_2}{v} e^{-i\xi_2} \\ \hat{w}_2 &= \frac{v_1}{v} e^{-i\xi_1} \end{aligned}$$

$$\begin{aligned} \kappa^D &= \frac{\sqrt{2}}{v} \text{diag}(m_d, m_s, m_b) \\ \kappa^U &= \frac{\sqrt{2}}{v} \text{diag}(m_u, m_c, m_t) \end{aligned}$$

CP conservation: CP is conserved if and only if:

$$\begin{aligned}
\operatorname{Im} J_1 &= \frac{1}{v^5}\sum_{i,j,k}\epsilon_{ijk}M_i^2 e_i e_j q_k = 0\\
\operatorname{Im} J_2 &= \frac{e_1e_2e_3}{v^9}\sum_{i,j,k}\epsilon_{ijk}M_i^4 M_k^2 = 0\\
\operatorname{Im} J_{30} &= \frac{1}{v^5}\sum_{i,j,k}\epsilon_{ijk}M_i^2 q_i e_j q_k = 0.
\end{aligned}$$

where J_i are the weak basis invariants found by Lavoura, Silva and Botella (1994, 1995) and discussed by Branco, Rebelo and Silva-Marcos (2005), Davidson, Gunion and Haber (2005), Ivanov (2006, 2007), Nishi (2006) and M. Maniatis, A. von Manteuffel and O. Nachtmann (2008).

CP conservation: The conditions for CP conservation could be rewritten as:

$$\begin{aligned}
\mathrm{Im}\, J_1 &= \frac{1}{v^5}\left[M_1^2 e_1(e_2 q_3 - e_3 q_2) + M_2^2 e_2(e_3 q_1 - e_1 q_3) + M_3^2 e_3(e_1 q_2 - e_2 q_1)\right] = 0 \\
\mathrm{Im}\, J_2 &= \frac{e_1 e_2 e_3}{v^9}(M_2^2 - M_1^2)(M_3^2 - M_2^2)(M_1^2 - M_3^2) = 0 \\
\mathrm{Im}\, J_{30} &= \frac{1}{v^5}\left[M_1^2 q_1(e_2 q_3 - e_3 q_2) + M_2^2 q_2(e_3 q_1 - e_1 q_3) + M_3^2 q_3(e_1 q_2 - e_2 q_1)\right] = 0
\end{aligned}$$

$$\mathrm{Im}\, J_1 = \frac{1}{v^5}\begin{vmatrix} q_1 & q_2 & q_3 \\ e_1 & e_2 & e_3 \\ e_1 M_1^2 & e_2 M_2^2 & e_3 M_3^2 \end{vmatrix}, \qquad \mathrm{Im}\, J_2 = \frac{2}{v^9}\begin{vmatrix} e_1 & e_2 & e_3 \\ e_1 M_1^2 & e_2 M_2^2 & e_3 M_3^2 \\ e_1 M_1^4 & e_2 M_2^4 & e_3 M_3^4 \end{vmatrix}$$

$$\mathrm{Im}\, J_{30} = \frac{1}{v^5}\begin{vmatrix} e_1 & e_2 & e_3 \\ q_1 & q_2 & q_3 \\ q_1 M_1^2 & q_2 M_2^2 & q_3 M_3^2 \end{vmatrix}$$

The H_1VV coupling as in the SM - the alignment limit

The LHC Higgs data imply that HZZ and HW^+W^- couplings are close to the SM prediction.

$$\Downarrow$$

$$H_iZ_\mu Z_\nu : \quad \frac{ig^2}{2\cos^2\theta_{\mathrm{W}}} e_i\, g_{\mu\nu}, \qquad H_iW^+_\mu W^-_\nu : \quad \frac{ig^2}{2} e_i\, g_{\mu\nu}$$

We define (within 2HDM) the alignment limit (AL) as $e_1 = v$

Then

$$e_1^2 + e_2^2 + e_3^2 = v^2 \qquad \Rightarrow \qquad e_2 = e_3 = 0$$

Note that no assumption has been made concerning the mass scale of beyond the SM bosons: M_2, M_3 and $M_{H^\pm}$.

M. Carena, I. Low, N. R. Shah, C. E. M. Wagner, "Impersonating the Standard Model Higgs Boson: Alignment without Decoupling", JHEP 1404 (2014) 015,

P. S. B. Dev, A. Pilaftsis, "Maximally Symmetric Two Higgs Doublet Model with Natural Standard Model Alignment", JHEP 1412 (2014) 024, Erratum: JHEP 1511 (2015) 147,

A. Pilaftsis, "Symmetries for standard model alignment in multi-Higgs doublet models", Phys.Rev. D93 (2016) no.7, 075012

Alignment conditions in terms of the potential parameters

$$\begin{aligned} v_2^2 \mathrm{Im}\,\left(e^{i\xi}\lambda_7\right) + v_2 v_1 \mathrm{Im}\,\left(e^{2i\xi}\lambda_5\right) + v_1^2 \mathrm{Im}\,\left(e^{i\xi}\lambda_6\right) &= 0, \\ v_2^4 \mathrm{Re}\,\left(e^{i\xi}\lambda_7\right) + v_2^3 v_1(-\lambda_2 + \lambda_{345}) + 3 v_2^2 v_1^2 \mathrm{Re}\,\left[e^{i\xi}(\lambda_6 - \lambda_7)\right] + & \\ + v_2 v_1^3 (\lambda_1 - \lambda_{345}) - v_1^4 \mathrm{Re}\,\left(e^{i\xi}\lambda_6\right) &= 0 \end{aligned}$$

where $\lambda_{345} \equiv \lambda_3 + \lambda_4 + \mathrm{Re}\,\left(e^{2i\xi}\lambda_5\right)$.

In the CP-conserving limit, with $\xi = 0$, $\mathrm{Im}\,\lambda_5 = \mathrm{Im}\,\lambda_6 = \mathrm{Im}\,\lambda_7 = 0$, we reproduce the single alignment condition found by Dev and Pilaftsis, JHEP 1412 (2014) 024,

If one wishes to satisfy the alignment conditions for any value of v_1, v_2 and ξ, the following constraints must be fulfilled:

$$\lambda_1 = \lambda_2 = \lambda_3 + \lambda_4, \;\; \lambda_5 = \lambda_6 = \lambda_7 = 0$$

- The above condition is inconsistent with CPV
- In a different context a potential that satisfies the above condition was considered in “Do precision electroweak constraints guarantee e^+e^- collider discovery of at least one Higgs boson of a two Higgs doublet model?” P. Chankowski, T. Farris, B.G, J. Gunion, J. Kalinowski, M. Krawczyk, Phys.Lett. B496 (2000) 195-205

Mixing angles in the alignment limit

The coupling of H_1 to a pair of vector bosons, e_1, could be written as follows:

$$e_1 = v\cos(\alpha_2)\cos(\alpha_1 - \beta)$$

The most general solution of the alignment condition: $e_1 = v$, $e_2 = 0$, $e_3 = 0$ reads

$$\alpha_2 = 0 \qquad \alpha_1 = \beta$$

The rotation matrix in this case becomes

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} c_\beta & s_\beta & 0 \\ -s_\beta\, c_3 & c_\beta\, c_3 & s_3 \\ s_\beta\, s_3 & -c_\beta\, s_3 & c_3 \end{pmatrix}.$$

The mixing matrix could be written in this case as

$$R = R_3 R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{pmatrix} \begin{pmatrix} c_\beta & s_\beta & 0 \\ -s_\beta & c_\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

CP-violation in the alignment limit requires the most general 2HDM

The J invariants in the alignment limit:

$$\begin{aligned}
\mathrm{Im}\, J_1 &= \frac{1}{v^5}\left[M_1^2 e_1(e_2q_3 - e_3q_2) + M_2^2 e_2(e_3q_1 - e_1q_3) + M_3^2 e_3(e_1q_2 - e_2q_1)\right] \to 0 \\
\mathrm{Im}\, J_2 &= \frac{e_1e_2e_3}{v^9}(M_2^2 - M_1^2)(M_3^2 - M_2^2)(M_1^2 - M_3^2) \to 0 \\
\mathrm{Im}\, J_{30} &= \frac{1}{v^5}\left[M_1^2 q_1(e_2q_3 - e_3q_2) + M_2^2 q_2(e_3q_1 - e_1q_3) + M_3^2 q_3(e_1q_2 - e_2q_1)\right] \\
&\to \frac{e_1q_2q_3}{v^3}(M_3^2 - M_2^2)
\end{aligned}$$

- Note that $e_1 = v$ implies no CP violation in H_iVV couplings ($\mathrm{Im}\, J_2 = 0$), the only possible CP violation may appear in cubic scalar couplings $H_2H^+H^-$ and $H_3H^+H^-$, proportional to q_2 and q_3, respectively.

- The necessary condition for CP violation is that both $H_2H^+H^-$ *and* $H_3H^+H^-$ must exist *together* with non-zero ZH_2H_3 vertex. The latter implies that for CP invariance either H_2 or H_3 would have to be odd under CP, on the other hand if *both* of them couple to H^+H^- (that is CP even), then there is no way to preserve CP.

In the case $\lambda_6 = \lambda_7 = 0$ the $(\mathcal{M}^2)_{13}$ and $(\mathcal{M}^2)_{23}$ are related as follows

$$(\mathcal{M}^2)_{13} = \tan\beta(\mathcal{M}^2)_{23}$$

As a consequence of the above relation there is a constraint that relates mass eigenvalues, mixing angles and $\tan\beta$ (Khater and Osland, 2003):

$$M_1^2 R_{13}(R_{12}\tan\beta - R_{11}) + M_2^2 R_{23}(R_{22}\tan\beta - R_{21}) + M_3^2 R_{33}(R_{32}\tan\beta - R_{31}) = 0$$

$$\Downarrow$$

Alignment limit $(\alpha_2 = 0, \alpha_1 = \beta) \Rightarrow$ $\boxed{(M_2^2 - M_3^2)s_3 c_3 s_\beta = 0}$

$$\Downarrow$$

- $M_2 \neq M_3$, but $\alpha_3 = 0, \pi/2$, then $q_3 = 0, q_2 = 0$, respectively (since in 2HDM5, in the alignment limit $\mathrm{Im}\,\lambda_5 = 0$) , so no CP violation, or

- $M_2 = M_3$, therefore $\mathrm{Im}\,J_3 = 0$, so again no CP violation.

Partial conclusions:

If $\lambda_6 = \lambda_7 = 0$, so within the $\mathbb{Z}_2$-symmetric model (2HDM5), the alignment $(e_1 = v, e_2 = e_3 = 0)$ implies no CP violation.

$$\Downarrow$$

In order to have scalar sector CPV in the alignment limit, the $\mathbb{Z}_2$ must be violated hardly (by dim 4 interactions).

$$\Downarrow$$

FCNC

Alignment with or without decoupling

Alignment limit: $e_1 = v, e_2 = e_3 = 0$

$$\begin{aligned} e_1^2 M_1^2 + e_2^2 M_2^2 + e_3^2 M_3^2 &= v_1^4\lambda_1 + v_2^4\lambda_2 + 2v_1^2 v_2^2(\lambda_3 + \lambda_4 + \mathrm{Re}\left[e^{2i\xi}\lambda_5\right]) \\ &\quad + 4v_1^3 v_2 \mathrm{Re}\left[e^{i\xi}\lambda_6\right] + 4v_1 v_2^3 \mathrm{Re}\left[e^{i\xi}\lambda_7\right] \end{aligned}$$

- If one requires the quartic coupling constants λ_i to remain in a perturbative regime, e.g. $\lambda_i < 4\pi$, then in the decoupling limit of $M_{2,3,H^\pm} \to \infty$, as its consequence, $e_{2,3} \to 0$ (so the SM is recovered as the low-energy effective theory, i.e. alignment with decoupling), so that $e_{2,3}^2 M_{2,3}^2 / v^4 \lesssim \mathcal{O}(1)$.

- If we had chosen $e_2 \sim e_3 \sim 0$ (AL), then any value of $M_{2,3,H^\pm} \gtrsim M_1$ would be allowed, in particular relatively light $H_{2,3,H^\pm}$ with $M_{2,3,H^\pm} \sim v$ would be a viable option (alignment without decoupling).

Alignment with or without decoupling

There is a physical distinction between the AL in the decoupling regime and the AL without decoupling:

- In either case, one must have $|e_2/v|$, $|e_3/v| \ll 1$, which means that the distinction between alignment with or without decoupling cannot be detected via the tree-level Higgs couplings to gauge bosons and fermions.

- However, the distinction is present in the cubic and quartic tree-level Higgs couplings, e.g. in $H_1H_1H_1$ coupling.

$$H_1H_1H_1 : \frac{M_1^2}{2v} - \frac{(e_2^2+e_3^2)M_{H^\pm}^2}{v^3} \sim \frac{M_1^2}{2v} - \begin{cases} \frac{M_{H^\pm}^2}{v}\left[\left(\frac{e_2}{v}\right)^2 + \left(\frac{e_3}{v}\right)^2\right] & \text{without decoupling} \\ e_2+e_3 & \text{with decoupling} \end{cases}$$

- Alignment without decoupling: $M^2_{2,3,H^\pm} \sim \mathcal{O}(v^2)$

- Alignment with decoupling (perturbativity): $e_{2,3}M^2_{2,3,H^\pm} \sim \mathcal{O}(v^3)$

"Heavy" Higgs bosons ($H_2, H_3, H^{\pm}$) in the alignment limit

Scalar-vector couplings in the alignment limit:

$$e_1 = v, \;\; e_2 = e_3 = 0$$

$$\boxed{H_1 Z_\mu Z_\nu : \;\; \frac{ig^2}{2\cos^2\theta_{\mathrm{W}}} v\, g_{\mu\nu}, \qquad H_1 W^+_\mu W^-_\nu \; : \;\; \frac{ig^2}{2} v\, g_{\mu\nu}}$$

$$\boxed{Z^\mu H_2 H_3 : \;\; \frac{g}{2v\cos\theta_{\mathrm{W}}}(p_2 - p_3)^\mu}$$

$$\begin{array}{ll} H_2 H^+ W^{-\mu} : \;\; \dfrac{ig}{2} e^{-i\alpha_3}(p_2 - p^+)^\mu, & H_2 H^- W^{+\mu} : \;\; \dfrac{-ig}{2} e^{i\alpha_3}(p_2 - p^-)^\mu, \\ H_3 H^+ W^{-\mu} : \;\; \dfrac{ig}{2} e^{-i(\alpha_3 + \pi/2)}(p_2 - p^+)^\mu, & H_3 H^- W^{+\mu} : \;\; \dfrac{-ig}{2} e^{i(\alpha_3 + \pi/2)}(p_2 - p^-)^\mu, \end{array}$$

Scalar couplings in the alignment limit:

Couplings between H_i and H^+H^- are given in the alignment limit (for $\xi = 0$) by:

$$
\begin{aligned}
q_1 &= \frac{1}{v}\left(2M_{H^\pm}^2 - 2\mu^2 + M_1^2\right) \\
q_2 &= +\, c_3\left[\frac{(c_\beta^2 - s_\beta^2)}{vc_\beta s_\beta}(M_2^2 - \mu^2) + \frac{v}{2s_\beta^2}\mathrm{Re}\,\lambda_6 - \frac{v}{2c_\beta^2}\mathrm{Re}\,\lambda_7\right] + s_3\frac{v}{2c_\beta s_\beta}\mathrm{Im}\,\lambda_5 \\
q_3 &= -\, s_3\left[\frac{(c_\beta^2 - s_\beta^2)}{vc_\beta s_\beta}(M_3^2 - \mu^2) + \frac{v}{2s_\beta^2}\mathrm{Re}\,\lambda_6 - \frac{v}{2c_\beta^2}\mathrm{Re}\,\lambda_7\right] + c_3\frac{v}{2c_\beta s_\beta}\mathrm{Im}\,\lambda_5
\end{aligned}
$$

$$
\begin{array}{llll}
H_1H_1H_1: & \dfrac{M_1^2}{2v} & H_jH_kH_k: & \dfrac{q_j}{2} \\[2ex]
H_1H_jH_j: & \dfrac{q_1}{2} + \dfrac{M_j^2 - M_{H^\pm}^2}{v} & H_iH^+H^-: & q_i
\end{array}
$$

where $i = 1, 2, 3$ and $j, k = 2, 3$.

Yukawa couplings in the alignment limit and fermionic Higgs boson decays:

A general (flavour non-diagonal) Yukawa coupling of Higgs H_α:

$$H_\alpha \bar{f}_k (A^{\alpha\, f}_{kl} + i\gamma_5 B^{\alpha\, f}_{kl}) f_l,$$

where $f = d, u$ with $\alpha = 1, 2, 3$ and $A^{\alpha\, f}$ and $B^{\alpha\, f}$ Hermitian matrices (as required by the Hermiticity of the Yukawa Lagrangian) in the flavour space with $k, l = 1, 2, 3$.

The following relations between scalar and pseudoscalar components of H_2 and H_3 Yukawa couplings hold in the alignment limit:

$$\begin{aligned} A^{2\,d}_{km} &= B^{3\,d}_{km}, & A^{2\,u}_{km} &= -B^{3\,u}_{km} \\ B^{2\,d}_{km} &= -A^{3\,d}_{km}, & B^{2\,u}_{km} &= A^{3\,u}_{km} \end{aligned}$$

The following sum rules are satisfied:

$$\begin{aligned} A^{2\,f}_{km} A^{2\,f}_{ij} + B^{2\,f}_{km} B^{2\,f}_{ij} &= A^{3\,f}_{km} A^{3\,f}_{ij} + B^{3\,f}_{km} B^{3\,f}_{ij} \\ A^{2\,f}_{km} B^{2\,f}_{ij} &= -A^{3\,f}_{km} B^{3\,f}_{ij}. \end{aligned}$$

The amount of CP violation (encoded by $A^{\alpha\, f}_{km} B^{\alpha\, f}_{ij}$) is, in the alignment limit, “opposite” for H_2 and H_3.

Yukawa couplings in the alignment limit and fermionic Higgs boson decays:

Diagonal up-type couplings:

$$\bar{u}_k u_k H_1 : \qquad -\frac{m_{u_k}}{v}$$

$$\bar{u}_k u_k H_2 : \qquad \frac{1}{\sqrt{2}}\left(-\mathrm{Re}\,\rho_k^U - i\gamma_5 \mathrm{Im}\,\rho_k^U\right)$$

$$\bar{u}_k u_k H_3 : \qquad \frac{1}{\sqrt{2}}\left(-\mathrm{Im}\,\rho_k^U + i\gamma_5 \mathrm{Re}\,\rho_k^U\right)$$

$$\Gamma(H_2 \to u_k \bar{u}_k) \propto M_2 \beta_k \left(|\mathrm{Re}\,\rho_k^U|^2 \beta_k^2 + |\mathrm{Im}\,\rho_k^U|^2\right),$$

with $\beta_k = \sqrt{1 - 4m_k^2/M_2^2}$. If $\beta_k \simeq 1$ then

$$\boxed{\frac{\Gamma(H_2 \to u_k \bar{u}_k)}{\Gamma(H_3 \to u_k \bar{u}_k)} = \frac{M_2}{M_3} + \mathcal{O}\left(|\rho_k^U|^2 \frac{{m_k^U}^2}{M_2^2}\right)}$$

Alignment-sensitive bosonic observables:

$$\frac{\Gamma(H_1 \to W^+W^-, ZZ)}{\Gamma(H_{\mathsf{SM}} \to W^+W^-, ZZ)} = 1 + \mathcal{O}\left(\frac{e_2^2}{v^2}, \frac{e_2 e_3}{v^2}, \frac{e_3^2}{v^2}\right)$$

$$\mathsf{BR}(H_{2,3} \to W^+W^-, ZZ, H_1 Z) = \mathcal{O}\left(\frac{e_2^2}{v^2}, \frac{e_2 e_3}{v^2}, \frac{e_3^2}{v^2}\right)$$

$$\mathsf{BR}(H_3 \to H_1 H_2) = \mathcal{O}\left(\frac{e_2^2}{v^2}, \frac{e_2 e_3}{v^2}, \frac{e_3^2}{v^2}\right)$$

$$\frac{\mathsf{BR}(H_3 \to H^+H^-)}{\mathsf{BR}(H_3 \to H_2 H_2)} = \sqrt{\frac{M_3^2 - 4M_{H^\pm}^2}{M_3^2 - 4M_2^2}}\left[1 - \frac{4(M_2^2 - M_{H^\pm}^2)}{q_3 v}\frac{e_3}{v} + \mathcal{O}\left(\frac{e_2^2}{v^2}, \frac{e_2 e_3}{v^2}, \frac{e_3^2}{v^2}\right)\right]$$

$$\frac{\mathsf{BR}(H^\pm \to H_2 W^\pm)}{\mathsf{BR}(H^\pm \to H_3 W^\pm)} = \frac{[\lambda(M_{H^\pm}, M_2, M_W)]^{3/2}}{[\lambda(M_{H^\pm}, M_3, M_W)]^{3/2}} + \mathcal{O}\left(\frac{e_2^2}{v^2}, \frac{e_2 e_3}{v^2}, \frac{e_3^2}{v^2}\right)$$

$$\frac{\mathsf{BR}(H_3 \to H_2 Z)}{\mathsf{BR}(H_3 \to H^+W^-)} = \frac{1}{c_W^2}\frac{[\lambda(M_3, M_2, M_Z)]^{3/2}}{[\lambda(M_3, M_{H^\pm}, M_W)]^{3/2}} + \mathcal{O}\left(\frac{e_2^2}{v^2}, \frac{e_2 e_3}{v^2}, \frac{e_3^2}{v^2}\right)$$

Spontaneous CP violation

The goal: to formulate conditions for SCPV in terms of observables

$$
\begin{aligned}
\mathrm{Im}\, J_1 &= \frac{1}{v^5}\sum_{i,j,k}\epsilon_{ijk}M_i^2 e_i e_k q_j \\
&= \frac{1}{v^5}[e_1e_2q_3(M_2^2-M_1^2)-e_1e_3q_2(M_3^2-M_1^2)+e_2e_3q_1(M_3^2-M_2^2)], \\
\mathrm{Im}\, J_2 &= \frac{2}{v^9}\sum_{i,j,k}\epsilon_{ijk}e_ie_je_kM_i^4M_k^2 = \frac{2e_1e_2e_3}{v^9}\sum_{i,j,k}\epsilon_{ijk}M_i^4M_k^2 \\
&= \frac{2e_1e_2e_3}{v^9}(M_2^2-M_1^2)(M_3^2-M_2^2)(M_3^2-M_1^2), \\
\mathrm{Im}\, J_{30} &\equiv \frac{1}{v^5}\sum_{i,j,k}\epsilon_{ijk}q_iM_i^2e_jq_k, \\
&= \frac{1}{v^5}[q_1q_2e_3(M_2^2-M_1^2)-q_1q_3e_2(M_3^2-M_1^2)+q_2q_3e_1(M_3^2-M_2^2)].
\end{aligned}
$$

Theorem: CP is conserved if and only if $\mathrm{Im}\, J_1 = \mathrm{Im}\, J_2 = \mathrm{Im}\, J_{30} = 0$.

$$
\begin{aligned}
V(\Phi_1, \Phi_2) = & -\frac{1}{2}\left\{m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 + \left[m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{H.c.}\right]\right\} \\
& + \frac{\lambda_1}{2}(\Phi_1^\dagger \Phi_1)^2 + \frac{\lambda_2}{2}(\Phi_2^\dagger \Phi_2)^2 + \lambda_3(\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_4(\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) \\
& + \frac{1}{2}\left[\lambda_5(\Phi_1^\dagger \Phi_2)^2 + \text{H.c.}\right] + \left\{\left[\lambda_6(\Phi_1^\dagger \Phi_1) + \lambda_7(\Phi_2^\dagger \Phi_2)\right](\Phi_1^\dagger \Phi_2) + \text{H.c.}\right\} \\
& \equiv Y_{a\bar{b}} \Phi_{\bar{a}}^\dagger \Phi_b + \frac{1}{2} Z_{a\bar{b}c\bar{d}} (\Phi_{\bar{a}}^\dagger \Phi_b)(\Phi_{\bar{c}}^\dagger \Phi_d).
\end{aligned}
$$

Theorem: In order for CP violation to be spontaneous, at least one of the $\mathrm{Im}\, J_i$ invariants must be non-zero, while four other weak-basis I invariants constructed from the coefficients of the potential, must vanish.

$$
\begin{aligned}
I_{Y3Z} &= \mathrm{Im}\left[Z_{a\bar{c}}^{(1)} Z_{e\bar{b}}^{(1)} Z_{b\bar{e}c\bar{d}} Y_{d\bar{a}}\right], \\
I_{2Y2Z} &= \mathrm{Im}\left[Y_{a\bar{b}} Y_{c\bar{d}} Z_{b\bar{a}d\bar{f}} Z_{f\bar{c}}^{(1)}\right], \\
I_{3Y3Z} &= \mathrm{Im}\left[Z_{a\bar{c}b\bar{d}} Z_{c\bar{e}d\bar{g}} Z_{e\bar{h}f\bar{q}} Y_{g\bar{a}} Y_{h\bar{b}} Y_{q\bar{f}}\right], \\
I_{6Z} &= \mathrm{Im}\left[Z_{a\bar{b}c\bar{d}} Z_{b\bar{f}}^{(1)} Z_{d\bar{h}}^{(1)} Z_{f\bar{a}j\bar{k}} Z_{k\bar{j}m\bar{n}} Z_{n\bar{m}h\bar{c}}\right].
\end{aligned}
$$

Theorem:
Let us assume that the quantity

$$D \equiv e_1^2 M_2^2 M_3^2 + e_2^2 M_3^2 M_1^2 + e_3^2 M_1^2 M_2^2$$

is non-zero. Then, in a charge-conserving general 2HDM, CP is violated spontaneously if and only if the following three statements are satisfied simultaneously:

- At least one of the three invariants $\mathrm{Im}\, J_1$, $\mathrm{Im}\, J_2$, $\mathrm{Im}\, J_{30}$ is nonzero.
- $M_{H^\pm}^2 = \dfrac{v^2}{2D}[e_1 q_1 M_2^2 M_3^2 + e_2 q_2 M_3^2 M_1^2 + e_3 q_3 M_1^2 M_2^2 - M_1^2 M_2^2 M_3^2]$,
- $q = \dfrac{1}{2D}[(e_2 q_3 - e_3 q_2)^2 M_1^2 + (e_3 q_1 - e_1 q_3)^2 M_2^2 + (e_1 q_2 - e_2 q_1)^2 M_3^2 + M_1^2 M_2^2 M_3^2]$.

Spontaneous CP violation in the alignment limit in terms of physical couplings

$$e_1 = v, e_2 = e_3 = 0$$

$$\Downarrow$$

$$\begin{aligned} M_{H^\pm}^2 &= \frac{vq_1 - M_1^2}{2}, \\ q &= \frac{1}{2}\left(\frac{q_2^2}{M_2^2} + \frac{q_3^2}{M_3^2} + \frac{M_1^2}{v^2}\right). \end{aligned}$$

Summary

- 2HDM allows for extra sources of CP-violation that might be useful to explain baryon asymmetry.
- In the alignment limit ($e_1 = v$, so that only H_1 couples to VV as in the SM) there is no CP-violation if $\lambda_6 = \lambda_7 = 0$ (softly broken $\mathbb{Z}_2$ imposed).
- The requirement of extra sources of CP-violation together with the alignment ($H_1 = H_{SM}$) favours the most general 2HDM with $\lambda_6 \neq 0$ and $\lambda_7 \neq 0$ (no $\mathbb{Z}_2$ symmetry).
- The requirement of extra sources of CP-violation together with the alignment ($H_1 = H_{SM}$) implies an interesting possibility of large FCNC that couple to Higgs bosons (in progress).
- There exists a set of alignment-sensitive observables for $H_2, H_3, H^\pm$ that might be useful to test the alignment scenario, e.g. BR$(H_{2,3} \to W^+W^-, ZZ, H_1H_1, H_1Z) =$

0, BR$(H_3 \to H_1 H_2) = 0$ or

$$\frac{\mathrm{BR}(H_3 \to H^+ H^-)}{\mathrm{BR}(H_3 \to H_2 H_2)} = \sqrt{\frac{M_3^2 - 4M_{H^\pm}^2}{M_3^2 - 4M_2^2}}$$

- In order to disprove SCPV a minimal set of measurements consists of $M_{H^\pm}$ and q_1, if they do not satisfy $q_1 = \left(2M_{H^\pm}^2 + M_1^2\right)/v$, then CP is not violated spontaneously.

- To prove SCPV is strictly speaking impossible since one would need to show that the conditions

$$\begin{aligned} M_{H^\pm}^2 &= \frac{v^2}{2D}[e_1 q_1 M_2^2 M_3^2 + e_2 q_2 M_3^2 M_1^2 + e_3 q_3 M_1^2 M_2^2 - M_1^2 M_2^2 M_3^2], \\ q &= \frac{1}{2D}[(e_2 q_3 - e_3 q_2)^2 M_1^2 + (e_3 q_1 - e_1 q_3)^2 M_2^2 + (e_1 q_2 - e_2 q_1)^2 M_3^2 + \\ &\quad + M_1^2 M_2^2 M_3^2]. \end{aligned}$$

hold exactly. Since measurements are always subject to experimental (and theoretical) uncertainties, indeed, the above equations could at best only hold within some confidence level. Note, however, that the verification of the above constraints requires a determination of 11 parameters. M_1 and v are already known, so 9 new measurements should be performed in order to test these constraints. Therefore we conclude that in order to test SCPV, all potential parameters must be known.

Backup slides

Stationary points of the potential

By demanding that the derivatives of the potential with respect to the fields should vanish in the vacuum, we end up with the following stationary-point equations:

$$
\begin{aligned}
m_{11}^2 &= v_1^2\lambda_1 + v_2^2(\lambda_3+\lambda_4) + \frac{v_2^2}{c_\xi}(\mathrm{Re}\,\lambda_5 c_\xi - \mathrm{Im}\,\lambda_5 s_\xi) \\
&\quad + \frac{v_1 v_2}{c_\xi}\left[\mathrm{Re}\,\lambda_6(2+c_{2\xi}) - \mathrm{Im}\,\lambda_6 s_{2\xi}\right] + \frac{v_2}{v_1 c_\xi}(v_2^2 \mathrm{Re}\,\lambda_7 - \mathrm{Re}\,m_{12}^2), \\
m_{22}^2 &= v_2^2\lambda_2 + v_1^2(\lambda_3+\lambda_4) + \frac{v_1^2}{c_\xi}(\mathrm{Re}\,\lambda_5 c_\xi - \mathrm{Im}\,\lambda_5 s_\xi) \\
&\quad + \frac{v_1 v_2}{c_\xi}\left[\mathrm{Re}\,\lambda_7(2+c_{2\xi}) - \mathrm{Im}\,\lambda_7 s_{2\xi}\right] + \frac{v_1}{v_2 c_\xi}(v_1^2 \mathrm{Re}\,\lambda_6 - \mathrm{Re}\,m_{12}^2), \\
\mathrm{Im}\,m_{12}^2 &= \frac{v_1 v_2}{c_\xi}(\mathrm{Re}\,\lambda_5 s_{2\xi} + \mathrm{Im}\,\lambda_5 c_{2\xi}) + \frac{v_1^2}{c_\xi}(\mathrm{Re}\,\lambda_6 s_\xi + \mathrm{Im}\,\lambda_6 c_\xi) \\
&\quad + \frac{v_2^2}{c_\xi}(\mathrm{Re}\,\lambda_7 s_\xi + \mathrm{Im}\,\lambda_7 c_\xi) - \mathrm{Re}\,m_{12}^2 t_\xi
\end{aligned}
$$

For completeness also the following relations are useful

$$
\begin{aligned}
M_1^2 + M_2^2 + M_3^2 &= \frac{v^2}{v_1 v_2 c_\xi} \mathrm{Re}\, m_{12}^2 + v_1^2 \lambda_1 + v_2^2 \lambda_2 - v^2 \mathrm{Re}\, \lambda_5 + v^2 t_\xi \mathrm{Im}\, \lambda_5 \\
&\quad - \frac{v_1 (v_1^2 - v_2^2 c_{2\xi})}{v_2 c_\xi} \mathrm{Re}\, \lambda_6 - 2 v_1 v_2 s_\xi \mathrm{Im}\, \lambda_6 \\
&\quad - \frac{v_2 (v_2^2 - v_1^2 c_{2\xi})}{v_1 c_\xi} \mathrm{Re}\, \lambda_7 - 2 v_1 v_2 s_\xi \mathrm{Im}\, \lambda_7, \qquad (3) \\
M_{H^\pm}^2 &= \frac{v^2}{2 v_1 v_2 c_\xi} \mathrm{Re}\, \left(m_{12}^2 - v_1^2 \lambda_6 - v_2^2 \lambda_7 - v_1 v_2 e^{i\xi} \left[\lambda_4 + \lambda_5 \right] \right).
\end{aligned}
$$

The above shows that for finite λ_i, increasing values of $M_{2,3}$ and $M_{H^\pm}$ require positive and increasing $\mathrm{Re}\, m_{12}^2$.

Alignment defined in terms of bilinear potential couplings and vevs

Using the minimization conditions the alignment conditions can be formulated as

$$\begin{aligned} \mathrm{Im}\, m_{12}^2 &= 0, \\ m_{11}^2 - m_{22}^2 &= \mathrm{Re}\, m_{12}^2 \left(\frac{v_1}{v_2} - \frac{v_2}{v_1} \right). \end{aligned}$$

$$\mathcal{M}^2_{11} = v_1^2\lambda_1 - v_2^2 s_\xi^2 \mathrm{Re}\,\lambda_5 - \frac{v_2^2}{2c_\xi} s_\xi c_{2\xi} \mathrm{Im}\,\lambda_5 + \frac{v_1 v_2}{2c_\xi}(1+2c_{2\xi})\mathrm{Re}\,\lambda_6$$
$$-2v_1 v_2 s_\xi \mathrm{Im}\,\lambda_6 - \frac{v_2^3}{2v_1 c_\xi}\mathrm{Re}\,\lambda_7 + \frac{v_2}{2v_1 c_\xi}\mathrm{Re}\,m_{12}^2,$$

$$\mathcal{M}^2_{22} = v_2^2\lambda_2 - v_1^2 s_\xi^2 \mathrm{Re}\,\lambda_5 - \frac{v_1^2}{2c_\xi} s_\xi c_{2\xi} \mathrm{Im}\,\lambda_5 + \frac{v_1 v_2}{2c_\xi}(1+2c_{2\xi})\mathrm{Re}\,\lambda_7$$
$$-2v_1 v_2 s_\xi \mathrm{Im}\,\lambda_7 - \frac{v_1^3}{2v_2 c_\xi}\mathrm{Re}\,\lambda_6 + \frac{v_1}{2v_2 c_\xi}\mathrm{Re}\,m_{12}^2,$$

$$\mathcal{M}^2_{33} = -v^2 c_\xi^2 \mathrm{Re}\,\lambda_5 - \frac{v^2}{2c_\xi}(2s_\xi^3 - 3s_\xi)\mathrm{Im}\,\lambda_5$$
$$-\frac{v^2 v_1}{2v_2 c_\xi}\mathrm{Re}\,\lambda_6 - \frac{v^2 v_2}{2v_1 c_\xi}\mathrm{Re}\,\lambda_7 + \frac{v^2}{2v_1 v_2 c_\xi}\mathrm{Re}\,m_{12}^2,$$

$$\mathcal{M}^2_{12} = v_1 v_2(\lambda_3+\lambda_4) + v_1 v_2 c_\xi^2 \mathrm{Re}\,\lambda_5 + \frac{v_1 v_2}{2c_\xi}(2s_\xi^3 - 3s_\xi)\mathrm{Im}\,\lambda_5 + \frac{v_1^2}{2c_\xi}(2+c_{2\xi})\mathrm{Re}\,\lambda_6$$
$$-v_1^2 s_\xi \mathrm{Im}\,\lambda_6 + \frac{v_2^2}{2c_\xi}(2+c_{2\xi})\mathrm{Re}\,\lambda_7 - v_2^2 s_\xi \mathrm{Im}\,\lambda_7 - \frac{1}{2c_\xi}\mathrm{Re}\,m_{12}^2,$$

$$\mathcal{M}^2_{13} = -\frac{1}{2}v v_2 s_{2\xi}\mathrm{Re}\,\lambda_5 - \frac{1}{2}v v_2 c_{2\xi}\mathrm{Im}\,\lambda_5 - v v_1 s_\xi \mathrm{Re}\,\lambda_6 - v v_1 c_\xi \mathrm{Im}\,\lambda_6,$$

$$\mathcal{M}^2_{23} = -\frac{1}{2}v v_1 s_{2\xi}\mathrm{Re}\,\lambda_5 - \frac{1}{2}v v_1 c_{2\xi}\mathrm{Im}\,\lambda_5 - v v_2 s_\xi \mathrm{Re}\,\lambda_7 - v v_2 c_\xi \mathrm{Im}\,\lambda_7.$$

The charge boson mass is given as follows

$$M_{H^\pm}^2 = \frac{v^2}{2v_1 v_2 c_\xi}\mathrm{Re}\left(m_{12}^2 - v_1^2\lambda_6 - v_2^2\lambda_7 - v_1 v_2 e^{i\xi}\left[\lambda_4 + \lambda_5\right]\right)$$

The eigenvalues of this matrix will be the masses of the three neutral scalars. In order to find these, a cubic equation needs to be solved. For our purposes, a different approach will suffice. We may rewrite the elements of the mass matrix $\mathcal{M}_{ij}^2$ in terms of the eigenvalues M_i^2 and elements of the rotation matrix R_{ij} as six equations:

$$\begin{aligned}
\mathcal{M}_{11}^2 &= M_1^2 R_{11}^2 + M_2^2 R_{21}^2 + M_3^2 R_{31}^2, && (4)\\
\mathcal{M}_{22}^2 &= M_1^2 R_{12}^2 + M_2^2 R_{22}^2 + M_3^2 R_{32}^2, && (5)\\
\mathcal{M}_{33}^2 &= M_1^2 R_{13}^2 + M_2^2 R_{23}^2 + M_3^2 R_{33}^2, && (6)\\
\mathcal{M}_{12}^2 &= M_1^2 R_{11}R_{12} + M_2^2 R_{21}R_{22} + M_3^2 R_{31}R_{32}, && (7)\\
\mathcal{M}_{13}^2 &= M_1^2 R_{11}R_{13} + M_2^2 R_{21}R_{23} + M_3^2 R_{31}R_{33}, && (8)\\
\mathcal{M}_{23}^2 &= M_1^2 R_{12}R_{13} + M_2^2 R_{22}R_{23} + M_3^2 R_{32}R_{33}. && (9)
\end{aligned}$$

The above seven equations are linear in the λ_i-parameters of the potential. We have 10 such parameters (counting both real and imaginary parts of λ_5, λ_6 and λ_7) and may now solve this set of seven equations for seven of the λ_i-parameters, thus expressing them in terms of the other parameters we have introduced. It is convenient to solve for the following set of parameters: $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mathrm{Re}\lambda_5, \mathrm{Im}\lambda_6, \mathrm{Im}\lambda_7)$. We also introduce the more convenient parameter μ^2 by putting

$$\mathrm{Re}\, m_{12}^2 = \frac{2v_1 v_2}{v^2}\mu^2.$$

Solving the set of equations, we arrive at

$$\begin{aligned}
\lambda_1 &= -\frac{v_2^2}{v^2 v_1^2 c_\xi^3}\mu^2 + \frac{\left(R_{11}v - R_{13}v_2 t_\xi\right)^2}{v^2 v_1^2}M_1^2 + \frac{\left(R_{21}v - R_{23}v_2 t_\xi\right)^2}{v^2 v_1^2}M_2^2 \\
&+ \frac{\left(R_{31}v - R_{33}v_2 t_\xi\right)^2}{v^2 v_1^2}M_3^2 - \frac{v_2^2 t_\xi}{2v_1^2 c_\xi^2}\mathrm{Im}\lambda_5 - \frac{v_2(2c_{2\xi}+1)}{2v_1 c_\xi^3}\mathrm{Re}\lambda_6 + \frac{v_2^3}{2v_1^3 c_\xi^3}\mathrm{Re}\lambda_7, \\
\lambda_2 &= -\frac{v_1^2}{v^2 v_2^2 c_\xi^3}\mu^2 + \frac{\left(R_{12}v - R_{13}v_1 t_\xi\right)^2}{v^2 v_2^2}M_1^2 + \frac{\left(R_{22}v - R_{23}v_1 t_\xi\right)^2}{v^2 v_2^2}M_2^2 \\
&+ \frac{\left(R_{32}v - R_{33}v_1 t_\xi\right)^2}{v^2 v_2^2}M_3^2 - \frac{v_1^2 t_\xi}{2v_2^2 c_\xi^2}\mathrm{Im}\lambda_5 + \frac{v_1^3}{2v_2^3 c_\xi^3}\mathrm{Re}\lambda_6 - \frac{v_1(2c_{2\xi}+1)}{2v_2 c_\xi^3}\mathrm{Re}\lambda_7, \\
\lambda_3 &= \frac{2}{v^2}M_{H^\pm}^2 - \frac{1}{v^2 c_\xi^3}\mu^2 + \frac{\left(R_{12}v - R_{13}v_1 t_\xi\right)\left(R_{11}v - R_{13}v_2 t_\xi\right)}{v^2 v_1 v_2}M_1^2 \\
&+ \frac{\left(R_{22}v - R_{23}v_1 t_\xi\right)\left(R_{21}v - R_{23}v_2 t_\xi\right)}{v^2 v_1 v_2}M_2^2 \\
&+ \frac{\left(R_{32}v - R_{33}v_1 t_\xi\right)\left(R_{31}v - R_{33}v_2 t_\xi\right)}{v^2 v_1 v_2}M_3^2 \\
&- \frac{1}{2c_\xi^2}t_\xi \mathrm{Im}\lambda_5 - \frac{v_1 c_{2\xi}}{2v_2 c_\xi^3}\mathrm{Re}\lambda_6 - \frac{v_2 c_{2\xi}}{2v_1 c_\xi^3}\mathrm{Re}\lambda_7,
\end{aligned}$$

$$
\begin{aligned}
\lambda_4 &= -\frac{2}{v^2}M_{H^\pm}^2 + \frac{c_{2\xi}}{v^2c_\xi^3}\mu^2 + \frac{R_{13}^2}{v^2c_\xi^2}M_1^2 + \frac{R_{23}^2}{v^2c_\xi^2}M_2^2 + \frac{R_{33}^2}{v^2c_\xi^2}M_3^2 \\
&\quad -\frac{1}{2c_\xi^2}t_\xi \mathrm{Im}\lambda_5 - \frac{v_1c_{2\xi}}{2v_2c_\xi^3}\mathrm{Re}\lambda_6 - \frac{v_2c_{2\xi}}{2v_1c_\xi^3}\mathrm{Re}\lambda_7, \\
\mathrm{Re}\lambda_5 &= \frac{1}{v^2c_\xi^3}\mu^2 - \frac{R_{13}^2}{v^2c_\xi^2}M_1^2 - \frac{R_{23}^2}{v^2c_\xi^2}M_2^2 - \frac{R_{33}^2}{v^2c_\xi^2}M_3^2 \\
&\quad +\frac{1}{4c_\xi^3}(3s_\xi + s_{3\xi})\mathrm{Im}\lambda_5 - \frac{v_1}{2v_2c_\xi^3}\mathrm{Re}\lambda_6 - \frac{v_2}{2v_1c_\xi^3}\mathrm{Re}\lambda_7, \\
\mathrm{Im}\lambda_6 &= -\frac{v_2t_\xi}{v^2v_1c_\xi^2}\mu^2 + \frac{R_{13}\left(R_{13}v_2t_\xi - R_{11}v\right)}{v^2v_1c_\xi}M_1^2 + \frac{R_{23}\left(R_{23}v_2t_\xi - R_{21}v\right)}{v^2v_1c_\xi}M_2^2 \\
&\quad +\frac{R_{33}\left(R_{33}v_2t_\xi - R_{31}v\right)}{v^2v_1c_\xi}M_3^2 - \frac{v_2}{2v_1c_\xi^3}\mathrm{Im}\lambda_5 - \frac{1}{2c_\xi^2}t_\xi c_{2\xi}\mathrm{Re}\lambda_6 + \frac{v_2^2t_\xi}{2v_1^2c_\xi^2}\mathrm{Re}\lambda_7, \\
\mathrm{Im}\lambda_7 &= -\frac{v_1t_\xi}{v^2v_2c_\xi^2}\mu^2 + \frac{R_{13}\left(R_{13}v_1t_\xi - R_{12}v\right)}{v^2v_2c_\xi}M_1^2 + \frac{R_{23}\left(R_{23}v_1t_\xi - R_{22}v\right)}{v^2v_2c_\xi}M_2^2 \\
&\quad +\frac{R_{33}\left(R_{33}v_1t_\xi - R_{32}v\right)}{v^2v_2c_\xi}M_3^2 - \frac{v_1}{2v_2c_\xi^3}\mathrm{Im}\lambda_5 + \frac{v_1^2t_\xi}{2v_2^2c_\xi^2}\mathrm{Re}\lambda_6 - \frac{1}{2c_\xi^2}t_\xi c_{2\xi}\mathrm{Re}\lambda_7.
\end{aligned}
$$