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PYTHAGOREAN SPINORS AND PENROSE TWISTORS

Andrzej Trautman

Instytut Fizyki Teoretycznej UW
Hoża 69, 00681 Warszawa, Poland
E-mail: amt@fuw.edu.pl

1 Introduction

Besides the major applications of spinors and twistors to equations of mathematical physics, there are minor results, where these objects play an auxiliary role or bring a new light on otherwise well-known facts. One example of such a result is the twistor-inspired derivation, based on the use of the conformal compactification of a (pseudo-) Euclidean space, of the fractional-linear form of Möbius transformations. Even simpler is the remark that the solution, attributed to Euclid, of the Pythagorean equation, has a spinorial interpretation: it is equivalent to the statement that a null vector in $\mathbb{Z}^3$, considered as a subset of $\mathbb{R}^3$ with a scalar product of signature $(1, 2)$, is the (tensor) square of an integer-valued spinor. In this short article, I expand the latter observation and give a summary of the rudiments of twistor notions associated with higher-dimensional spaces; this account is included here not because of its novelty, but in the hope that there may be some interest in a presentation by an outsider. Only 'global' twistors are considered here; I gave a brief account of my view of 'local' twistors in (Trautman 1993).

Originally, Roger Penrose intended twistor spaces to be associated with, or serve as replacements for, the 4-dimensional, Lorentzian space-times. His belief in the privileged and exceptional role of four dimensions, apparent in twistor theory, was strikingly confirmed by the discoveries of exotic differential structures on $\mathbb{R}^4$, and of the Donaldson and Seiberg-Witten invariants. There are, however, interesting generalizations of twistor ideas to other dimensions and signatures; especially to proper Riemannian 3- and 4-manifolds. As is often the case with important ideas, the original notion of twistor has been generalized in many ways; only some of them are briefly presented below.

2 Pythagorean spinors

If p and q are integers, then the triple of integers (x, y, z), given by

$$(1) \qquad x = p^2 - q^2, \quad y = 2pq, \quad z = p^2 + q^2,$$

is *Pythagorean*: it satisfies the equation $x^2+y^2 = z^2$. If (x, y, z) is Pythagorean, then at least one of the numbers x and y is even; moreover, if $t \in \mathbb{Z}$, then (y, x, z) and (tx, ty, tz) are also Pythagorean. I say that a Pythagorean triple (x, y, z) is *standard* if $z > 0$ and either the triple (x, y, z) is relatively prime (rp) and y is even or $(x/2, y/2, z/2)$ is a triple of rp integers and $y/2$ is odd. For example, the triples $(-1, 0, 1)$ and $(8, 6, 10)$ are standard, but $(4, 3, 5)$ and $(6, 8, 10)$ are not. Every Pythagorean triple can be written as (tx, ty, tz), where $t \in \mathbb{Z}$, the integers (x, y, z) are rp and $z > 0$; if y is even, then (x, y, z) is standard; if y is odd, then $(2x, 2y, 2z)$ is standard.

Proposition 1. *If (x, y, z) is a standard Pythagorean triple, then there is a pair (p, q) of relatively prime integers such that* (1) *holds.*

In other words: there is a bijection between the set of directions in $\mathbb{Z}^2$ and the set of 'null directions' in $\mathbb{Z}^3$.

Proof. Note that $z > 0$, y even and $y^2 = (z+x)(z-x)$ imply $z+x = 2m \geq 0$ and $z - x = 2n \geq 0$, where m and n are integers. If $y = 2r$, then the

Pythagorean equation is equivalent to $r^2 = mn$. If the triple (x, y, z) is *rp*, then so is the triple (m, n, r). If r is odd and the integers x, y and z are all even, but have no divisor > 2, then the triple (m, n, r) is also *rp*. If (m, n, r) is *rp*, then $r^2 = mn$ implies that both m and n are squares. $\square$

Recall the classical lemma (Sierpiński 1987):

Lemma 1. If p and q are integers, then a necessary and sufficient condition for the existence of integers u and v, such that $pu + qv = 1$, is that p and q be relatively prime.

It leads to

Proposition 2. *The group* $\mathsf{SL}_2(\mathbb{Z})$ *acts transitively on the set* $P \subset \mathbb{Z}^2$ *of integer-valued 'spinors' with relatively prime components.*

In other words: the group $\mathsf{SL}_2(\mathbb{Z})$ acts transitively on the set of directions in $\mathbb{Z}^2$.

Proof. Indeed, let $a, b, c, d \in \mathbb{Z}$ and consider

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2}$$

The matrix A is in $\mathsf{SL}_2(\mathbb{Z})$ iff $ad - bc = 1$; it acts in $\mathbb{Z}^2$ by sending $\begin{pmatrix} p \\ q \end{pmatrix}$ to $\begin{pmatrix} p' \\ q' \end{pmatrix} = A \begin{pmatrix} p \\ q \end{pmatrix}$. If $\begin{pmatrix} p \\ q \end{pmatrix} \in P$, then there are integers u and v such that $pu+qv = 1$. Putting $(u', v') = (u, v)A^{-1}$ one obtains $p'u'+q'v' = 1$; therefore, $\mathsf{SL}_2(\mathbb{Z})$ acts in P. This action is transitive: if $\begin{pmatrix} p \\ q \end{pmatrix} \in P$ and $pu + qv = 1$, then the matrix $\begin{pmatrix} p & -v \\ q & u \end{pmatrix}$ sends $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} p \\ q \end{pmatrix}$. $\square$

The stabilizer of an element of P is a subgroup of $\mathsf{SL}_2(\mathbb{Z})$ isomorphic to $\mathbb{Z}$.

Recall that the group $\mathsf{SL}_2(\mathbb{R})$ is the connected component of the group $\mathsf{Spin}_{1,2}(\mathbb{R})$: there is the exact sequence of homomorphisms of groups,

$$1 \to \mathbb{Z}_2 \to \mathsf{SL}_2(\mathbb{R}) \xrightarrow{\rho} \mathsf{SO}^{\circ}_{1,2}(\mathbb{R}) \to 1.$$

If $(x, y, z) \in \mathbb{R}^3$ is represented by the matrix $\begin{pmatrix} z+x & y \\ y & z-x \end{pmatrix} = 2\begin{pmatrix} p \\ q \end{pmatrix}\begin{pmatrix} p & q \end{pmatrix}$ and

$$v = \begin{pmatrix} z+x & y \\ y & z-x \end{pmatrix} J, \text{ where } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3}$$

so that $v^2 = (x^2+y^2-z^2)I$, then $\rho(A)v = AvA^{-1}$. By restriction, one obtains the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathsf{SL}_2(\mathbb{Z}) \xrightarrow{\rho} \mathsf{G} \to 1.$$

The group $\mathsf{G} \subset \mathsf{SO}^{\mathrm{o}}_{1,2}(\mathbb{R})$ which, by definition, is the image of $\mathsf{SL}_2(\mathbb{Z})$ by ρ, is a group of matrices with entries that are either integer or half-integer. For example,

$$\rho\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -1 & 1 & 1 \\ -\frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix}.$$

The group G acts on the set $\{(x, y, z) \in \mathbb{Z}^3 : x + z \text{ is even}\}$. It is an easy exercise to find the subgroup of $\mathsf{SL}_2(\mathbb{Z})$ that covers the subgroup of G containing all matrices with integer elements.

3 Projective quadrics and twistors

Global twistors, described in the first paper on the subject (Penrose 1967), are associated with *projective quadrics*, i.e. with conformal compactifications of (pseudo-) Euclidean spaces. Most of the time, the name 'projective quadric' is shortened to 'quadric'. Let V be an m-dimensional vector space over $K = \mathbb{R}$ or $\mathbb{C}$ with a non-degenerate quadratic form g. In the vector space $W = V \oplus K^2$ one introduces the quadratic form h by putting $h(w) = g(v) + \lambda\mu$, where $w = (v, \lambda, \mu) \in W$, and defines the quadric $\mathsf{Q}(h)$ to be the subset of the projective space $\mathsf{P}(W)$ consisting of all null directions, $\mathsf{Q}(h) = \{\mathrm{dir} w : w \in W, w \neq 0, h(w) = 0\}$. The quadric inherits from the quadratic space (W, h) a (locally flat) *conformal structure*. The map $V \to \mathsf{Q}(h)$, given by $v \mapsto \mathrm{dir}(v, -g(v), 1)$, is a conformal embedding with an image open and dense in $\mathsf{Q}(h)$; the complement of the image is the 'null cone at infinity'. The group $\mathsf{Spin}(h)$ acts on $\mathsf{Q}(h)$ by sending $\mathrm{dir} w$ to $\mathrm{dir}(AwA^{-1})$, where $w \in W$ and $A \in \mathsf{Spin}(h)$. This action is conformal and transitive. The Clifford algebra $\mathsf{Cl}(h)$, associated with the quadratic space (W, h), is isomorphic to the algebra $\mathsf{Cl}(g) \otimes \mathsf{End}\, K^2$; an isomorphism is induced by the Clifford map

$$(v, \lambda, \mu) \mapsto \begin{pmatrix} v & \lambda \\ \mu & -v \end{pmatrix} = X. \tag{4}$$

Therefore, every $A \in \mathsf{Spin}(h) \subset \mathsf{Cl}(h)$ can be written in the form (2), where a, b, c and d are now suitable elements of $\mathsf{Cl}(g)$. Let $a \mapsto {}^t a$ be the antiautomorphism of the algebra $\mathsf{Cl}(g)$ such that ${}^t 1 = 1$ and ${}^t v = v$ for every $v \in V$. There holds

Proposition 3. *If $A \in \mathsf{Spin}(h)$, then the space*

$$V_A = \{v \in V : cv + d \ \text{ is invertible in } \ \mathsf{Cl}(g)\}$$

is open and dense in V; the map

$$f_A : V_A \to V_{A^{-1}} \ \text{ defined by } \ f_A(v) = (av + b)(cv + d)^{-1}$$

is a conformal diffeomorphism, $g(\mathrm{d}v) = {}^t(cv + d)(cv + d) g(\mathrm{d} f_A(v))$.

This result goes back to Th. Vahlen; see (Robinson and Trautman 1993) and the references given there. The Clifford algebra $\mathsf{Cl}(g)$ has an irreducible Dirac (m even) or Pauli (m odd) representation γ in a complex, $2^{[\frac{1}{2}m]}$-dimensional vector space S of spinors. The Dirac representation, restricted to the even Clifford algebra, decomposes into the sum of two Weyl representations in the spaces of spinors of opposite 'chirality'. A representation γ of $\mathsf{Cl}(g)$ in S extends to a representation δ of $\mathsf{Cl}(h)$ in $S \oplus S$. Namely, if $a \in \mathsf{Cl}(g)$ and $b \in \mathsf{End}\, K^2 \subseteq \mathsf{End}\, \mathbb{C}^2$, then

$$\delta : \mathsf{Cl}(h) \to \mathsf{End}\,(S \oplus S) = (\mathsf{End}\, S) \otimes \mathsf{End}\, \mathbb{C}^2 \ \text{ is given by } \ \delta(a \otimes b) = \gamma(a) \otimes b. \tag{5}$$

3.1 The complex case

3.1.1 Complex projective quadrics of dimension m

Assume $K = \mathbb{C}$ so that $V = \mathbb{C}^m$. The corresponding complex quadric Q_m is, in the words of Kobayashi and Ochiai (1982), 'a holomorphic analogue of a sphere'. It has no complex-bilinear Riemannian structure; its complex conformal structure supports a unique conformal spin structure which can be described as follows. Let Cl_m denote the Clifford algebra associated with $(\mathbb{C}^m, g)$. The conformal spin (Clifford) group is defined here as the subset Cpin_m of Cl_m consisting of products of all *even* sequences of non-null vectors in V. This group acts on vectors by sending, for every $a \in \mathsf{Cpin}_m$, the vector v to $\rho(a)v = av\,{}^t\!a$. There is the exact sequence of group homomorphisms,

$$1 \to \mathsf{K}_m \to \mathsf{Cpin}_m \xrightarrow{\rho} \mathsf{CO}_m \to 1,$$

where CO_m is the connected component of the conformal group (understood here as the group of rotations and dilations). If m is odd, then the kernel K_m of ρ is $\mathbb{Z}_2 = \{1, -1\}$. For m even, ρ gives a four-fold cover and $\mathsf{K}_m = \{1, -1, \eta, -\eta\}$, where $\eta \in \mathsf{Cl}_m$ is a volume element normalized so that ${}^t\eta\eta = -1$. The group K_m is isomorphic to $\mathbb{Z}_4$ for $m \equiv 0 \bmod 4$ and to $\mathbb{Z}_2 \times \mathbb{Z}_2$ for $m \equiv 2 \bmod 4$. The spin group is $\mathsf{Spin}(g) = \mathsf{Spin}_m = \{a \in \mathsf{Cpin}_m : {}^t a a = 1\}$. The group Spin_{m+2} acts transitively on Q_m. The image of the null vector $w_\infty = (0, 1, 0) \in W$ by (4) is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathsf{Cl}_{m+2}$. The stabilizer of $\mathrm{dir} w_\infty \in \mathsf{Q}_m$ is the semi-direct product H_m of Cpin_m by $\mathbb{C}^m$ given explicitly by

$$\mathsf{H}_m = \left\{ \begin{pmatrix} a & v \\ 0 & {}^t a^{-1} \end{pmatrix} : a \in \mathsf{Cpin}_m,\ v \in \mathbb{C}^m \right\} \subset \mathsf{Spin}_{m+2}\,.$$

The groups Cpin_m and $\mathbb{C}^m$ are thus made into subgroups of Spin_{m+2}. Let PO_m be the quotient of the complex special orthogonal group SO_m by its centre: $\mathsf{PO}_{2n+1} = \mathsf{SO}_{2n+1}$ and $\mathsf{PO}_{2n} = \mathsf{SO}_{2n}/\mathbb{Z}_2$. There is the commutative diagram

$$\begin{array}{ccc} \mathsf{Cpin}_m & \longrightarrow & \mathsf{CO}_m \\ \big\downarrow & & \big\downarrow \\ \mathsf{Spin}_{m+2} & \longrightarrow & \mathsf{PO}_{m+2} \end{array}$$

of group homomorphisms: the vertical arrows are injective and the horizontal ones are 4:1 or 2:1 depending on whether m is even or odd. The map $\mathbb{C}^m \to \mathsf{Spin}_{m+2}$ descends to a monomorphism of groups, $\mathbb{C}^m \to \mathsf{PO}_{m+2}$. With these observations in mind, one can formulate

Proposition 4. (i) *The conformal spin structure on* Q_m *is given by the principal bundle maps*

$$\begin{array}{ccccc} \mathsf{Cpin}_m & \longrightarrow & \mathsf{CO}_m & & \\ \big\downarrow & & \big\downarrow & & \\ \mathsf{Spin}_{m+2}/\mathbb{C}^m & \longrightarrow & \mathsf{PO}_{m+2}/\mathbb{C}^m & \longrightarrow & \mathsf{Q}_m\,. \end{array}$$

(ii) *The associated bundle of spinors,*

$$(\mathsf{Spin}_{m+2}/\mathbb{C}^m) \times_{\mathsf{Cpin}_m} S \to \mathsf{Q}_m,$$

corresponding to the representation $\gamma : \mathsf{Cpin}_m \to \mathsf{GL}(S)$, *is isomorphic to the bundle* $\Sigma \to \mathsf{Q}_m$, *where*

$$\Sigma = \{(\mathrm{dir}w, \Phi) \in \mathsf{Q}_m \times S \oplus S : \delta(w)\Phi = 0\}$$

and δ *is as in* (5).
(iii) *The Maurer-Cartan form* $A^{-1}\mathrm{d}A$ *defines a flat Cartan connection on the* H_m*-bundle* $\varpi : \mathsf{Spin}_{m+2} \to \mathsf{Q}_m$.

A proof of (i) is in (Robinson and Trautman 1993). The map $\varpi : \mathsf{Spin}_{m+2} \to \mathsf{Q}_m$ is given by $\varpi(A) = \mathrm{dir}(Aw_\infty A^{-1})$. Part (ii) generalizes a similar observation made by Manin (1981) for $m = 4$. I learned of this generalization from Harnad; the isomorphism in question is given by $[(A\mathbb{C}^m, \varphi)] \mapsto (\varpi(A), \delta(A)(\varphi, 0))$. Part (iii) follows from the definition of a Cartan connection; see (Friedrich 1977) and the references given there.

If $\Phi \in S \oplus S$ is a non-zero spinor, then the vector space $\{w \in W : \delta(w)\Phi = 0\}$ is totally null; if it is maximal (*mtn*), then Φ is said to be *pure.* If $m = 2n$ is even, then a pure spinor is Weyl (chiral) and the $(n+1)$-vector formed from a linear basis spanning the corresponding *mtn* is either self-dual or antiself-dual. The projective *twistor space* T_m for Q_m is the manifold of directions of pure spinors associated with (W, h). For m even, it has two components, T^+_m and T^-_m. If one puts $m = 2n$ (m even) or $m = 2n-1$ (m odd), then $\dim \mathsf{T}_m = \frac{1}{2}n(n+1)$. In particular, each of the spaces T_3, T^+_4 and T^-_4 is diffeomorphic to $\mathbb{C}\mathsf{P}_3$. A global twistor $\mathrm{dir}\Phi \in \mathsf{T}_m$ is identified with the *mtn* space of vectors annihilating Φ. This space descends to a totally null geodesic submanifold of Q_m of the maximal dimension $[\frac{1}{2}m]$. The dimensions of Q_m and T_m coincide only for $m = 3$ (minitwistors; cf. the papers by K. P. Tod in (Mason *et al.* 1995); see also (Ward 1996) and the papers by N. J. Hitchin referred to there) and $m = 6$ (in this case, the three spaces Q_6, T^+_6 and T^-_6 are diffeomorphic to each other; this coincidence reflects triality; cf. the papers by L. P. Hughston in (Mason *et al.* 1995)). The flag manifold for Q_m is defined as the 'projectivized' bundle of pure spinors, $\mathsf{F}_m = \{(\mathrm{dir}w, \mathrm{dir}\Phi) \in \mathsf{Q}_m \times \mathsf{T}_m : \delta(w)\Phi = 0\}$. The two natural projections define the double fibration $\mathsf{Q}_m \leftarrow \mathsf{F}_m \to \mathsf{T}_m$ which underlies the *Penrose correspondence* (Wells 1979). For m even, F_m has two connected components and there are two such double fibrations.

3.1.2 The case of four dimensions

Instead of representing W as $V \oplus \mathbb{C}^2$, one uses, in this case, the identification of $\mathbb{C}^6$ with $\wedge^2\mathbb{C}^4$. Let $\mathbb{T}$ be the complex, four-dimensional vector space of *Penrose twistors*; $\mathbb{T}$ is assumed to be endowed with a volume element $\varepsilon \in \wedge^4\mathbb{T}^*$, $\varepsilon \neq 0$. A frame $(e_\alpha)_{\alpha=1,\dots,4}$ in $\mathbb{T}$ is said to be *unimodular* if $\varepsilon = e^1 \wedge e^2 \wedge e^3 \wedge e^4$, where (e^α) is the frame in $\mathbb{T}^*$, dual to (e_α). From now on, only unimodular frames are used. The six-dimensional vector space $W = \wedge^2\mathbb{T}$ has a quadratic form h—the *Pfaffian*—defined by $\frac{1}{2}w \wedge w = h(w)e_1 \wedge e_2 \wedge e_3 \wedge e_4$. The volume element defines also the Hodge map $\star : \wedge\mathbb{T} \to \wedge\mathbb{T}^*$, such that $\star(1_{\wedge\mathbb{T}}) = \varepsilon$. If $w = \frac{1}{2}w^{\alpha\beta}e_\alpha \wedge e_\beta$, then $\star w = \frac{1}{2}\star w_{\alpha\beta}e^\alpha \wedge e^\beta$, where $\star w_{12} = w^{34}$, etc. If $w \in W$ is considered as a linear map $\mathbb{T}^* \to \mathbb{T}$ and $\star w$ as a linear map $\mathbb{T} \to \mathbb{T}^*$, then

$$w \circ \star w = -h(w)\mathrm{id}_{\mathbb{T}} \quad \text{and} \quad \star w \circ w = -h(w)\mathrm{id}_{\mathbb{T}^*}. \tag{6}$$

In the notation with indices, these equations read $w^{\alpha\gamma} \star w_{\gamma\beta} = -\delta^\alpha_\beta(w^{12}w^{34} + w^{13}w^{42} + w^{14}w^{23})$. The *Klein quadric* is $\mathsf{Q}_4 = \{\mathrm{dir} w : w \in W, w \neq 0, w \wedge w = 0\}$. By (6), the linear map $W \to \mathsf{End}(\mathbb{T} \oplus \mathbb{T}^*)$ given by

$$w \mapsto \begin{pmatrix} 0 & w \\ \star w & 0 \end{pmatrix} \tag{7}$$

has the Clifford property and yields a faithful and irreducible representation of $\mathsf{Cl}(h) = \mathsf{Cl}_6$ in $\mathbb{T} \oplus \mathbb{T}^*$. With respect to this representation, the elements of $\mathbb{T}$ and $\mathbb{T}^*$ are Weyl spinors of opposite chirality; using the notation of §3.1.1 one can put $\mathsf{T}_4^+ = \mathsf{P}(\mathbb{T})$ and $\mathsf{T}_4^- = \mathsf{P}(\mathbb{T}^*)$. The projective twistor $\mathrm{dir}\Phi$, $0 \neq \Phi \in \mathbb{T}$, is identified with the *mtn* 3-space $\{w \in W : w \wedge \Phi = 0\}$; this space projects to a totally null, geodesic, self-dual 2-dimensional submanifold of Q_4: $\alpha(\Phi) = \{\mathrm{dir}(\Phi \wedge \Phi') : \Phi' \in \mathbb{T}, \Phi \wedge \Phi' \neq 0\}$. As a complex manifold, $\alpha(\Phi)$ is $\mathbb{C}\mathsf{P}_2$. If $\Phi, \Phi' \in \mathbb{T}$ and $\Phi \wedge \Phi' \neq 0$, then $\mathrm{dir}(\Phi \wedge \Phi') \in \mathsf{Q}_4$ is the intersection of $\alpha(\Phi)$ and $\alpha(\Phi')$. Similarly, if $\Psi \in \mathbb{T}^*$, $\Psi \neq 0$, then there is the submanifold of Q_4: $\beta(\Psi) = \{\mathrm{dir}(\Phi \wedge \Phi') : \Phi, \Phi' \in \mathbb{T}, \Phi \wedge \Phi' \neq 0, \langle\Phi, \Psi\rangle = \langle\Phi', \Psi\rangle = 0\}$. The submanifolds $\alpha(\Phi)$ and $\beta(\Psi)$ intersect along a null geodesic iff $\langle\Phi, \Psi\rangle = 0$; as a complex manifold, such a null geodesic is $\mathbb{C}\mathsf{P}_1$; two distinct points $\mathrm{dir}w$, $\mathrm{dir}w' \in \mathsf{Q}_4$ lie on such a null geodesic iff $w \wedge w' = 0$; see §9.3 in (Penrose and Rindler 1986) and (Penrose 1996).

The group $\mathsf{Spin}(h) = \mathsf{Spin}_6$ is isomorphic to $\mathsf{SL}_4 = \mathsf{SL}(\mathbb{T})$ embedded in

$\mathsf{Cl}(h)$ by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix} \tag{8}$$

where $A^* \in \mathsf{SL}(\mathbb{T}^*)$ is the transpose of $A \in \mathsf{SL}(\mathbb{T})$. The element A acts in W by sending w to AwA^*, as may be checked from (7), (8) and the equation $\star(AwA^*) = (\det A)(A^{*-1} \star wA^{-1})$ valid for every $w \in W$ and $A \in \mathsf{GL}(\mathbb{T})$. A frame (e_α) in $\mathbb{T}$ can be used to construct a 'null frame' $(w_a)_{a=0,1,\ldots,4,\infty}$ in W by putting (say): $w_0 = e_3 \wedge e_4$, $w_1 = e_1 \wedge e_3$, $w_2 = e_1 \wedge e_4$, $w_3 = e_2 \wedge e_3$, $w_4 = e_2 \wedge e_4$ and $w_\infty = e_1 \wedge e_2$. For $z = (z^\mu) \in \mathbb{C}^4$, put $w(z) = w_0 + z^\mu w_\mu + (z_1 z_4 - z_2 z_3) w_\infty$; then for every z one has $w(z) \neq 0$ and $w(z) \wedge w(z) = 0$; the map $z \mapsto \operatorname{dir} w(z)$ is a conformal embedding of $V = \mathbb{C}^4$ in Q_4. Put $S = \operatorname{span}\{e_1, e_2\}$ and $S' = \operatorname{span}\{e_3, e_4\}$; the direction of w_∞ (equivalently: the plane S) is preserved by the subgroup

$$\mathsf{H}_4 = \{\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} : a \in \mathsf{GL}(S),\ b \in \mathsf{GL}(S'),\ \det a \det b = 1 \ \text{ and } \ v \in \mathsf{Hom}(S', S)\}$$

of SL_4 so that Cpin_4 is isomorphic to $\{(a, b) \in \mathsf{GL}_2 \times \mathsf{GL}_2 : \det a \det b = 1\}$.

3.2 The real case

Assume now $K = \mathbb{R}$ and let (k, l), $k + l = m$, be the signature of g. The real quadric $\mathsf{Q}_{k,l}$ is diffeomorphic to $(\mathbb{S}_k \times \mathbb{S}_l)/\mathbb{Z}_2$. In particular, $\mathsf{Q}_{k,0} = \mathbb{S}_k$; a proper real quadric, i.e. one with $kl \neq 0$, is orientable iff $k + l$ is even (Cahen *et al.* 1993). An essential difference between the complex and the real case is that, in the latter, the conformal structure on the quadric is generated by a pseudo-Riemannian metric. One can consider spin or pin structures corresponding to such a metric. (S)pin structures on real quadrics have been determined and a method for finding the spectrum of the Dirac operator given in (Cahen *et al.* 1995). There is neither room nor need to describe here the construction of the twistor spaces associated with the real quadrics. The most important case of $\mathsf{Q}_{1,3}$ is fully treated in the works of Penrose and his school. Instead, I describe here the *real twistors* on $\mathsf{Q}_{1,2}$ that could have been discovered by Euclid, had he followed the 'spinorial method' of solving the Pythagorean equation, outlined in §2.

3.2.1 Real twistors on $\mathsf{Q}_{1,2}$

Let $\lambda, \mu \in \mathbb{R}$ and let v be as in (3). The matrix X, given by (4), can be now considered as an endomorphism of $\mathbb{U}$, a four-dimensional vector space of *real* twistors. The antisymmetric matrix

$$\omega = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} : \mathbb{U} \to \mathbb{U}^*$$

is a symplectic 2-form on $\mathbb{U}$ and $\varepsilon = \frac{1}{2}\omega \wedge \omega$ is the corresponding volume 4-form. It follows from $X^* = \omega \circ X \circ \omega^{-1}$ that the map $X \circ \omega^{-1} : \mathbb{U}^* \to \mathbb{U}$ is antisymmetric. Since $X^2 = (x^2+y^2-z^2+\lambda\mu)\mathrm{id}_{\mathbb{U}}$, if the vector $(x, y, z, \lambda, \mu) \in \mathbb{R}^6$ is null, then the bivector $X \circ \omega^{-1}$ is of rank ≤ 2 and there are twistors $\Phi, \Psi \in \mathbb{U}$ such that $X \circ \omega^{-1} = \Phi \wedge \Psi$. Moreover, $\mathrm{tr}X = 0$ implies $\omega(\Phi, \Psi) = 0$. Conversely, given a four-dimensional real symplectic space $(\mathbb{U}, \omega)$, the vector space $W = \{w \in \wedge^2\mathbb{U} : \mathrm{tr}(w \circ \omega) = 0\}$ is five-dimensional and the restriction of the Pfaffian to W is a quadratic form of signature (2,3). Therefore, the quadric $\mathsf{Q}_{1,2}$ can be identified with the set of null directions in W, or, equivalently, with the set of *lagrangian planes* in $\mathbb{U}$. A real twistor $\Phi \in \mathbb{U}$ defines the null geodesic $\gamma(\Phi) = \{\mathrm{dir}(\Phi \wedge \Psi) : \Psi \in \mathbb{U},\ \omega(\Phi, \Psi) = 0\}$ on $\mathsf{Q}_{1,2}$. If $\Phi \wedge \Psi \neq 0$ and $\omega(\Phi, \Psi) = 0$, then $\gamma(\Phi) \cap \gamma(\Psi) = \mathrm{dir}(\Phi \wedge \Psi)$. Two distinct points of $\mathsf{Q}_{1,2}$ lie on one null geodesic iff the corresponding lagrangian planes intersect along a line. For the material of this paragraph, see Note 1 to Chapter 6 in (Woodhouse 1980) and §7.2 in (Penrose and Rindler 1986).

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