Unbounded linear operators

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Contents

Unbounded operators is a relatively technical and complicated subject. To my knowledge, in most mathematics departments of the world it does not belong to the standard curriculum, except maybe for some rudimentary elements. Most courses of functional analysis limit themselves to bounded operators, which are much cleaner and easier to discuss.

Of course, in physics departments unbounded operators do not belong to the standard curriculum either. However, implicitly, they appear very often in physics courses. In fact, many operators relevant for applications are unbounded.

These lecture notes grew out of a course "Mathematics of quantum theory" given at Faculty of Physics, University of Warsaw. The aim of the course was not only to give a general theory of unbounded operators, but also to illustrate it with many interesting examples.

Chapter 1

Unbounded operators on Banach spaces

1.1 Relations

One of the problems with unbounded operators is that they are not true operators. In order to avoid confusion, it is helpful to begin with a reexamination the concepts of functions and relations.

Let X, Y be sets. R is called a relation iff $R \subset Y \times X$. We will also write $R : X \to Y$.

(Note the inversion of the direction). An example of a relation is the identity

$$\mathbb{1}_X := \{(x,x) \ : \ x \in X\} \subset X \times X.$$

Introduce the projections

$$Y \times X \ni (y,x) \mapsto \pi_Y(y,x) := y \in Y,$$

$$Y \times X \ni (y,x) \mapsto \pi_X(y,x) := x \in X,$$

and the flip

$$Y \times X \ni (y,x) \mapsto \tau(y,x) := (x,y) \in X \times Y.$$

The domain of R is defined as $\mathrm{Dom}\, R := \pi_X R$, its range is $\mathrm{Ran}\, R = \pi_Y R$, the inverse of R is defined as $R^{-1} := \tau R \subset X \times Y$. If $S \subset Z \times Y$, then the superposition of S and R is defined as

$$S \circ R := \{(z,x) \in Z \times X \ : \ \exists_{y \in Y} \ (z,y) \in S, \ (y,x) \in R\}.$$

If $X_0 \subset X$, then the restriction of R to X_0 is defined as

$$R\Big|_{X_0} := R \cap Y \times X_0.$$

If, moreover, $Y_0 \subset Y$, then

$$R\Big|_{X_0 \to Y_0} := R \cap Y_0 \times X_0.$$

We say that a relation R is injective, if $\pi_X(R \cap \{y\} \times X)$ is one-element for any $y \in \operatorname{Ran} R$. We say that R is surjective if $\operatorname{Ran} R = Y$.

We say that a relation R is coinjective, if $\pi_Y(R \cap Y \times \{x\})$ is one-element for any $x \in \operatorname{Dom} R$. We say that R is cosurjective if $\operatorname{Dom} R = X$.

Proposition 1.1.1 *a) If R, S are coinjective, then so is $S \circ R$.*
b) If R, S are cosurjective, then so does $S \circ R$.

In a basic course of set theory we learn that a coinjective cosurjective relation is called a function. One also introduce many synonims of this word, such as a transformation, operator, map, etc.

To speak about ubounded operators we will need a more general concept. A coinjective relation will be called a partial transformation (or a partial operator, etc).

We also introduce the graph of R:

$$\mathrm{Gr}\, R := \{(x, y) \in X \times Y \ : \ (y, x) \in R\}.$$

Strictly speaking $\mathrm{Gr}\, R = \tau R$. The difference between $\mathrm{Gr}\, R$ and R lies only in their syntactic role.

Note that the order $Y \times X$ is convenient for the definition of superposition. However, it is not the usual choice. In the sequel, instead of writing $(y, x) \in R$, we will write $y = R(x)$ or $(x, y) \in \mathrm{Gr}\, R$.

A superposition of partial transformations is a partial transformation. The inverse of a partial transformation is a partial transformation iff it is injective.

A transformation (sometimes also called a total transformation) is a cosurjective partial transformation. The composition of transformations is a transformation.

We say that a transformation R is bijective iff it is injective and surjective. The inverse of a transformation is a transformation iff it is bijective.

Proposition 1.1.2 *Let $R \subset X \times Y$ and $S \subset Y \times X$ be transformations such that $R \circ S = \mathbb{1}_Y$ and $S \circ R = \mathbb{1}_X$. Then S and R are bijections and $S = R^{-1}$.*

1.2 Linear partial operators

Let $\mathcal{X}, \mathcal{Y}$ be vector spaces.

Proposition 1.2.1 (1) *A linear subspace $\mathcal{V} \subset \mathcal{X} \oplus \mathcal{Y}$ is a graph of a certain partial operator iff $(0, y) \in \mathcal{V}$ implies $y = 0$.*

(2) *A linear partial operator A is injective iff $(x, 0) \in \mathrm{Gr}\, A$ implies $x = 0$.*

From now on by an "operator" we will mean a "linear partial operator". To say that $A : \mathcal{X} \to \mathcal{Y}$ is a true operator we will write $\mathrm{Dom}\, A = \mathcal{X}$ or that it is everywhere defined.

For linear operators we will write Ax instead of $A(x)$ and AB instead of $A \circ B$. We define the kernel of an operator A:

$$\mathrm{Ker} A := \{x \in \mathrm{Dom}\, A \ : \ Ax = 0\}.$$

Suppose that A, B are two operators $\mathcal{X} \to \mathcal{Y}$. Then by $A + B$ we will mean the obvious operator with domain $\mathrm{Dom}\, A \cap \mathrm{Dom}\, B$.

1.3 Closed operators

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. Recall that $\mathcal{X} \oplus \mathcal{Y}$ can viewed as a Banach space equipped with a norm

$$\|(x, y)\|_1 := \|x\| + \|y\|.$$

Actually, we can use also any other norm p on $\mathbb{R}^2$ and replace this with $p(\|x\|, \|y\|)$. In particular, in the case of Hilbert spaces it is more appropriate to use the norm

$$\|(x, y)\|_2 := \sqrt{\|x\|^2 + \|y\|^2}.$$

Anyway, all these norms are equivalent and the convergence $(x_n, y_n) \to (x, y)$ is equivalent to $x_n \to x$, $y_n \to y$.

Theorem 1.3.1 *Let $A : \mathcal{X} \to \mathcal{Y}$ be an operator. The following conditions are equivalent:*

(1) $\mathrm{Gr}\, A$ *is closed in* $\mathcal{X} \oplus \mathcal{Y}$.

(2) *If* $x_n \to x,\ x_n \in \mathrm{Dom}\, A$ *and* $Ax_n \to y$, *then* $x \in \mathrm{Dom}\, A$ *and* $y = Ax$.

(3) $\mathrm{Dom}\, A$ *with the norm*

$$\|x\|_A := \|x\| + \|Ax\|.$$

is a Banach space.

Proof. The equivalence of (1), (2) and (3) is obvious, if we note that

$$\mathrm{Dom}\, A \ni x \mapsto (x, Ax) \in \mathrm{Gr}\, A$$

is a bijection. □

Definition 1.3.2 *An operator satisfying the above conditions is called closed.*

Theorem 1.3.3 *If A is closed and injective, then so is A^{-1}.*

Proof. The flip $\tau : \mathcal{X} \oplus \mathcal{Y} \to \mathcal{Y} \oplus \mathcal{X}$ is continuous. $\square$

Proposition 1.3.4 *If A is a closed operator, then $\mathrm{Ker} A$ is closed.*

1.4 Bounded operators

We will say that $A : \mathcal{X} \to \mathcal{Y}$ is bounded iff there exists c such as

$$\|Ax\| \leq c\|x\|. \tag{1.4.1}$$

The infimum of c on the right of (1.4.1) is called the norm of A and is denoted by $\|A\|$. In other words,

$$\|A\| := \sup_{\|x\|=1,\ x \in \mathrm{Dom}\, A} \|Ax\|. \tag{1.4.2}$$

$B(\mathcal{X}, \mathcal{Y})$ will denote all bounded everywhere defined operators from $\mathcal{X}$ to $\mathcal{Y}$.

Proposition 1.4.1 *A bounded operator A is closed iff $\mathrm{Dom}\, A$ is closed.*

If $A : \mathcal{X} \to \mathcal{Y}$ is closed, then $A \in B(\mathrm{Dom}\, A, \mathcal{Y})$.

Let us quote without a proof a well known theorem:

Theorem 1.4.2 (Closed graph theorem) *Let $A : \mathcal{X} \to \mathcal{Y}$ be a closed operator with $\mathrm{Dom}\, A = \mathcal{X}$. Then A is bounded.*

Proposition 1.4.3 *Let ξ be a densely defined linear form. The following conditions are equivalent:*

(1) *ξ is closed.*

(2) *ξ is everywhere defined and bounded.*

(3) *ξ is everywhere defined and $\mathrm{Ker}\xi$ is closed.*

1.5 Closable operators

Theorem 1.5.1 *Let $A : \mathcal{X} \to \mathcal{Y}$ be an operator. The following conditions are equivalent:*

(1) *There exists a closed operator B such that $B \supset A$.*

(2) $(\mathrm{Gr}\, A)^{\mathrm{cl}}$ *is the graph of an operator.*

(3) $(0, y) \in (\mathrm{Gr}\, A)^{\mathrm{cl}} \;\Rightarrow\; y = 0$.

(4) $(x_n) \subset \mathrm{Dom}\, A$, $x_n \to 0$, $Ax_n \to y$ *implies* $y = 0$.

Definition 1.5.2 *An operator A satisfying the conditions of Theorem 1.5.1 is called closable. If the conditions of Theorem 1.5.1 hold, then the operator whose graph equals $(\mathrm{Gr}\, A)^{\mathrm{cl}}$ is denoted by A^{cl} and called the closure of A.*

Proof of Theorem 1.5.1 To show (2)⇒(1) it suffices to take as B the operator A^{cl}. Let us show (1)⇒(2). Let B be a closed operator such that $A \subset B$. Then $(\mathrm{Gr}\, A)^{\mathrm{cl}} \subset (\mathrm{Gr}\, B)^{\mathrm{cl}} = \mathrm{Gr}\, B$. But $(0, y) \in \mathrm{Gr}\, B \;\Rightarrow\; y = 0$, hence $(0, y) \in (\mathrm{Gr}\, A)^{\mathrm{cl}} \;\Rightarrow\; y = 0$. Thus $(\mathrm{Gr}\, A)^{\mathrm{cl}}$ is the graph of an operator. □

As a by-product of the above proof, we obtain

Proposition 1.5.3 *If A is closable, B closed and $A \subset B$, then $A^{\mathrm{cl}} \subset B$.*

Proposition 1.5.4 *Let A be bounded. Then A is closable,* $\mathrm{Dom}\, A^{\mathrm{cl}} = (\mathrm{Dom}\, A)^{\mathrm{cl}}$ *and* $\|A^{\mathrm{cl}}\| = \|A\|$.

Proposition 1.5.5 *If A is a closable operator, then* $(\mathrm{Ker} A)^{\mathrm{cl}} \subset \mathrm{Ker} A^{\mathrm{cl}}$

Example 1.5.6 *Let $\mathcal{V}$ be a subspace in $\mathcal{X}$ and $x_0 \in \mathcal{X} \backslash \mathcal{V}$. Define the linear functional w such that $\mathrm{Dom}\, w = \mathcal{V} + \mathbb{C}x_0$, $\mathrm{Ker} w = \mathcal{V}$ and $\langle w|x_0\rangle = 1$. Then w is closable iff $x_0 \notin \mathcal{V}^{\mathrm{cl}}$. In particular, if $\mathcal{V}$ is dense, then w is nonclosable.*

1.6 Essential domains

Let A be a closed operator. We say that a linear subspace $\mathcal{D}$ is an essential domain of A iff $\mathcal{D}$ is dense in $\mathrm{Dom}\, A$ in the graph topology. In other words, $\mathcal{D}$ is an essential domain for A, if

$$\left(A\Big|_{\mathcal{D}}\right)^{\mathrm{cl}} = A.$$

Theorem 1.6.1 (1) *If $A \in B(\mathcal{X}, \mathcal{Y})$, then a linear subspace $\mathcal{D} \subset \mathcal{X}$ is an essential domain for A iff it is dense in $\mathcal{X}$ (in the usual topology).*

(2) *If A is closed, has a dense domain and $\mathcal{D}$ is its essential domain, then $\mathcal{D}$ is dense in $\mathcal{X}$.*

(2) follows from the following fact:

Proposition 1.6.2 *Let $\mathcal{V} \subset \mathcal{X}$ be Banach spaces with $\|x\|_{\mathcal{X}} \leq \|x\|_{\mathcal{V}}$. Then a dense subspace in $\mathcal{V}$ is dense in $\mathcal{X}$.*

1.7 Perturbations of closed operators

Definition 1.7.1 *Let B, $A : \mathcal{X} \to \mathcal{Y}$. We say that B is bounded relatively to A iff $\mathrm{Dom}\, A \subset \mathrm{Dom}\, B$ and there exist constants a, b such that*

$$\|Bx\| \leq a\|Ax\| + b\|x\|, \quad x \in \mathrm{Dom}\, A. \tag{1.7.3}$$

The infimum of a satisfying (1.7.3) is called the A-bound of B. If $\mathrm{Dom}\, A \not\subset \mathrm{Dom}\, B$ the A-bound of B is set $+\infty$.

In other words: the A-bound of B equals

$$a_1 := \inf_{c>0} \sup_{x \in \mathrm{Dom}\, A \setminus \{0\}} \frac{\|Bx\|}{\|Ax\| + c\|x\|}.$$

In particular, if B is bounded, then its A-bound equals 0.

If A is unbounded, then its A-bound equals 1.

In the case of Hilbert spaces it is more convenient to use the following condition to define the relative boundedness:

Theorem 1.7.2 *The A-bound of B equals*

$$a_1 = \inf_{c>0} \sup_{x \in \mathrm{Dom}\, A \setminus \{0\}} \left(\frac{\|Bx\|^2}{\|Ax\|^2 + c\|x\|^2} \right)^{1/2} . \tag{1.7.4}$$

Proof. For any $\epsilon > 0$ we have

$$\begin{aligned} \left(\|Ax\|^2 + c^2\|x\|^2\right)^{\frac{1}{2}} &\leq \|Ax\| + c\|x\| \\ &\leq \left((1+\epsilon^2)\|Ax\|^2 + c^2(1+\epsilon^{-2})\|x\|^2\right)^{\frac{1}{2}} . \end{aligned}$$

$\square$

Theorem 1.7.3 *Let A be closed and let B be bounded relatively to A with the A-bound less than 1. Then $A+B$ with the domain $\mathrm{Dom}\, A$ is closed. All essential domains of A are essential domains of $A+B$.*

Proof. We know that

$$\|Bx\| \le a\|Ax\| + b\|x\|$$

for some $a < 1$ and b. Hence

$$\|(A+B)x\| + \|x\| \le (1+a)\|Ax\| + (1+b)\|x\|$$

and

$$(1-a)\|Ax\| + \|x\| \le \|Ax\| - \|Bx\| + (1+b)\|x\| \le \|(A+B)x\| + (1+b)\|x\|.$$

Hence the norms $\|Ax\| + \|x\|$ and $\|(A+B)x\| + \|x\|$ are equivalent on $\operatorname{Dom} A$. $\square$

In particular, every bounded operator with domain containing $\operatorname{Dom} A$ is bounded relatively to A.

Proposition 1.7.4 *Suppose that $\mathcal{X} = \mathcal{Y}$. Then we have the following seemingly different* *definition of the A-bound of B:*

$$a_1 := \inf_{\mu \in \mathbb{C}} \inf_{c>0} \sup_{x \in \mathrm{Dom}\, A \setminus \{0\}} \frac{\|Bx\|}{\|(A-\mu)x\| + c\|x\|}.$$

Proof. It suffices to note that

$$\|Ax\| + c\|x\| \leq \|(A-\mu)x\| + (\mu + c)\|x\|.$$

□

Theorem 1.7.5 *Suppose that A, C are two operators with the same domain $\mathrm{Dom}\, A = \mathrm{Dom}\, C = \mathcal{D}$ satisfying*

$$\|(A-C)x\| \leq a(\|Ax\| + \|Cx\|) + b\|x\|$$

for some $a < 1$. Then

(1) *A is closed on $\mathcal{D}$ iff C is closed on $\mathcal{D}$.*

(2) *$\mathcal{D}$ is an essential domain of $A^{\rm cl}$ iff it is an essential domain of $C^{\rm cl}$.*

Proof. Define $B := C - A$ and $F(t) := A + tB$ with the domain $\mathcal{D}$. For $0 \leq t \leq 1$, we have

$$\begin{aligned} \|Bx\| &\leq a(\|Ax\| + \|Cx\|) + b\|x\| \\ &= a\left(\|(F(t) - tB)x\| + \|(F(t) + (1-t)B)x\|\right) + b\|x\| \\ &\leq 2a\|F(t)x\| + a\|Bx\| + b\|x\| \end{aligned}$$

Hence

$$\|Bx\| \leq \frac{2a}{1-a}\|F(t)x\| + \frac{b}{1-a}\|x\|.$$

Therefore, if $|s| < \frac{1-a}{2a}$ and $t, t+s \in [0,1]$, then $F(t+s)$ is closed iff $F(t)$ is closed. $\square$

1.8 Invertible unbounded operators

Let A be an operator from $\mathcal{X}$ to $\mathcal{Y}$.

Definition 1.8.1 *We say that an operator A is invertible (or boundedly invertible) iff $A^{-1} \in$*

$B(\mathcal{Y},\mathcal{X})$.

Note that we do not demand that A be densely defined. However, if A is invertible, then necessarily $\mathrm{Ran}\, A = \mathcal{Y}$.

The following criterion for the invertibility is obvious:

Proposition 1.8.2 *Let $C \in B(\mathcal{Y},\mathcal{X})$ be such that $\mathrm{Ran}\, C \subset \mathrm{Dom}\, A$ and $AC = \mathbb{1}$. Then A is invertible and $C = A^{-1}$.*

Theorem 1.8.3 (Closed range theorem) *Let A be closed. Suppose that for some $c > 0$*

$$\|Ax\| \geq c\|x\|, \quad x \in \mathrm{Dom}\, A. \tag{1.8.5}$$

Then $\mathrm{Ran}\, A$ is closed. If $\mathrm{Ran}\, A = Y$, then A is invertible and

$$\|A^{-1}\| \leq c^{-1}. \tag{1.8.6}$$

Proof. Let $y_n \in \mathrm{Ran}\, A$ and $y_n \to y$. Let $Ax_n = y_n$. Then x_n is a Cauchy sequence. Hence there exists $\lim_{n\to\infty} x_n := x$. But A is closed, hence $Ax = y$. Therefore, $\mathrm{Ran}\, A$ is closed. □

Corollary 1.8.4 *For an operator A, suppose that for some $c > 0$ (1.8.5) holds.*

(1) *Let A be closable. Then (1.8.5) holds for A^{cl} as well.*

(2) *Let A be closed and $\operatorname{Ran} A$ be dense in $\mathcal{Y}$. Then A is invertible and $\|A^{-1}\| \leq c^{-1}$.*

Theorem 1.8.5 *Let A be invertible and $\operatorname{Dom} B \supset \operatorname{Dom} A$.*

(1) *B has the A-bound less than $\|BA^{-1}\|$.*

(2) *If $\|BA^{-1}\| < 1$, then $A + B$ with the domain $\operatorname{Dom} A$ is closed, invertible and*

$$(A+B)^{-1} = \sum_{j=0}^{\infty} (-1)^j A^{-1} (BA^{-1})^j.$$

Proof. By the estimate

$$\|Bx\| \leq \|BA^{-1}\| \|Ax\|, \quad x \in \operatorname{Dom} A,$$

we see that B has the A-bound less than or equal to $\|BA^{-1}\|$. This proves (1).

Assume now that $\|BA^{-1}\| < 1$. Let

$$C_n := \sum_{j=0}^{n} (-1)^j A^{-1} (BA^{-1})^j.$$

Then $\lim_{n\to\infty} C_n =: C$ exists.

Let $y \in \mathcal{Y}$. Clearly, $\lim_{n\to\infty} C_n y = Cy$.

$$(A+B)C_n y = y + (-1)^n (BA^{-1})^{n+1} y \to y.$$

But $A+B$ is closed, hence $Cy \in \mathrm{Dom}(A+B)$ and $(A+B)Cy = y$. By Prop. 1.8.2, $A+B$ is invertible and $C = (A+B)^{-1}$. □

Theorem 1.8.6 *Let A and C be invertible and $\mathrm{Dom}\, C \supset \mathrm{Dom}\, A$. Then*

$$C^{-1} - A^{-1} = C^{-1}(A-C)A^{-1}.$$

Proposition 1.8.7 (1) *Let $B : \mathcal{X} \to \mathcal{Y}$ be closed and bounded. Let $A : \mathcal{Y} \to \mathcal{Z}$ be closed.*

Then AB is closed.

(2) *Let $C : \mathcal{Y} \to \mathcal{Z}$ be closed and invertible. Let $A : \mathcal{X} \to \mathcal{Y}$ be closed. Then CA is closed.*

1.9 Spectrum of unbounded operators

Let A be an operator on $\mathcal{X}$. We define the resolvent set of A as

$$\mathrm{rs}A := \{z \in \mathbb{C} \ : \ z\mathbb{1} - A \ \text{ is invertible } \}.$$

We define the spectrum of A as $\mathrm{sp}A := \mathbb{C}\backslash \mathrm{rs}A$.

We say that $x \in \mathcal{X}$ is an eigenvector of A with eigenvalue $\lambda \in \mathbb{C}$ iff $x \in \mathrm{Dom}\, A$, $x \neq 0$ and $Ax = \lambda x$. The set of eigenvalues is called the point spectrum of A and denoted $\mathrm{sp}_{\mathrm{p}}A$. Clearly, $\mathrm{sp}_{\mathrm{p}}A \subset \mathrm{sp}A$.

Let $\mathbb{C} \cup \{\infty\}$ denote the Riemann sphere (the one-point compactification of $\mathbb{C}$). The extended resolvent set is defined as $\mathrm{rs}^{\mathrm{ext}}A := \mathrm{rs}A \cup \{\infty\}$ if $A \in B(\mathcal{X})$ and $\mathrm{rs}^{\mathrm{ext}}A := \mathrm{rs}A$, if

A is unbounded. The extended spectrum is defined as

$$\mathrm{sp}^{\mathrm{ext}} A = \mathbb{C} \cup \{\infty\} \backslash \mathrm{rs}^{\mathrm{ext}} A.$$

If $A \in B(\mathcal{X})$, we set $(\infty - A)^{-1} = 0$.

Theorem 1.9.1 (1) *If* $\mathrm{rs} A$ *is nonempty, then* A *is closed.*

(2) *If* $z_0 \in \mathrm{rs} A$, *then* $\Big\{z : \ |z - z_0| < \|(z_0 - A)^{-1}\|^{-1}\} \subset \mathrm{rs} A$.

(3) $\|(z - A)^{-1}\| \geq (\mathrm{dist}(z, \mathrm{sp} A))^{-1}$.

(4) *If* A *is bounded, then* $\{|z| > \|A\|\}$ *is contained in* $\mathrm{rs} A$.

(5) $\mathrm{sp}^{\mathrm{ext}} A$ *is a compact subset of* $\mathbb{C} \cup \{\infty\}$.

(6) *If* $\lambda, \mu \in \mathrm{rs} A$, *then*

$$(z_1 - A)^{-1} - (z_2 - A)^{-1} = (z_2 - z_1)(z_1 - A)^{-1}(z_2 - A)^{-1}.$$

(7) *If* $z \in \mathrm{rs} A$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}z}(z - A)^{-1} = -(z - A)^{-2}.$$

(8) $(z-A)^{-1}$ *is analytic on* $\mathrm{rs}^{\mathrm{ext}}A$.

(9) $(z-A)^{-1}$ *cannot be analytically extended to a larger subset of* $\mathbb{C}\cup\{\infty\}$ *than* $\mathrm{rs}^{\mathrm{ext}}(A)$.

(10) $\mathrm{sp}^{\mathrm{ext}}(A)\neq\emptyset$

(11) $\mathrm{Ran}\,(z-A)^{-1}$ *does not depend on* $z\in\mathrm{rs}A$ *and equals* $\mathrm{Dom}\,A$.

(12) $\mathrm{Ker}(z-A)^{-1}=\{0\}$.

Proof. (1): If $\lambda\in\mathrm{rs}(A)$, then $\lambda-A$ is invertible, hence closed. $\lambda-A$ is closed iff A is closed.

(2): For $|z-z_0|<\|(z_0-A)^{-1}\|^{-1}$, we have $\|(z-z_0)(z_0-A)^{-1}\|<1$ Hence we can apply Theorem 2.

By (2), $\mathrm{dist}(z_0,\mathrm{sp}A)\geq\|(z_0-A)^{-1}\|^{-1}$. This implies (3).

(4): We check that $\sum\limits_{n=0}^{\infty}z^{-n-1}A^n$ is convergent for $|z|>\|A\|$ and equals $(z-A)^{-1}$.

(5): By (2), $\mathrm{sp}^{\mathrm{ext}}A\cap\mathbb{C}=\mathrm{sp}A$ is closed in $\mathbb{C}$. For bounded A, $\mathrm{sp}^{\mathrm{ext}}A$ is bounded by (4). For unbounded A, $\infty\in\mathrm{sp}^{\mathrm{ext}}A$. So in both cases, $\mathrm{sp}^{\mathrm{ext}}A$ is closed inin $\mathbb{C}\cap\{\infty\}$.

(6) follows from Thm 1.8.6. Note that it implies the continuity of the resolvent.

(7) follows from (6).

(8) follows from (7).

(9) follows from (3).

(10): For bounded A, $(z - A)^{-1}$ is an analytic function tending to zero at infinity. Hence it cannot be analytic everywhere, unless it is zero, which is impossible. For unbounded A, $\infty \in \mathrm{sp}^{\mathrm{ext}}A$.

(11) and (12) follow from (6). $\Box$

Proposition 1.9.2 *Suppose that* $\mathrm{rs}A$ *is non-empty and* $\mathrm{Dom}\,A$ *is dense. Then* $\mathrm{Dom}\,A^2$ *is dense.*

Proof. Let $z \in \mathrm{rs}A$. $(z - A)^{-1}$ is a bounded operator with a dense range and $\mathrm{Dom}\,A$ is dense. Hence $(z-A)^{-1}\,\mathrm{Dom}\,A$ is dense. $A(z-A)^{-1}\,\mathrm{Dom}\,A = (z-A)^{-1}A\,\mathrm{Dom}\,A \subset \mathrm{Dom}\,A$ Hence $(z - A)^{-1}\,\mathrm{Dom}\,A \subset \mathrm{Dom}\,A^2$. $\Box$

Theorem 1.9.3 *Let* A *and* B *be operators on* $\mathcal{X}$ *with* $A \subset B$, $A \neq B$. *Then* $\mathrm{rs}A \subset \mathrm{sp}B$, *and hence* $\mathrm{rs}B \subset \mathrm{sp}A$.

Proof. Let $\lambda \in \mathrm{rs}A$. Let $x \in \mathrm{Dom}\, B \backslash \mathrm{Dom}\, A$. We have $\mathrm{Ran}\,(\lambda - A) = \mathcal{X}$, hence there exists $y \in \mathrm{Dom}\, A$ such that $(\lambda - A)y = (\lambda - B)x$. Hence $(\lambda - B)y = (\lambda - B)x$. Hence $\lambda \notin \mathrm{rs}B$. $\square$

1.10 Functional calculus

Let $K \subset \mathbb{C} \cup \{\infty\}$ be compact. By $\mathrm{Hol}(K)$ let us denote the set of analytic functions on a neighborhood of K. It is a commutative algebra.

More precisely, let $\widetilde{\mathrm{Hol}}(K)$ be the set of pairs $(f, \mathcal{D})$, where $\mathcal{D}$ is an open subset of $\mathbb{C} \cup \{\infty\}$ containing K and f is an analytic function on $\mathcal{D}$. We introduce the relation $(f_1, \mathcal{D}_1) \sim (f_2, \mathcal{D}_2)$ iff $f_1 = f_2$ on a neighborhood of K contained $\mathcal{D}_1 \cap \mathcal{D}_2$. We set $\mathrm{Hol}(K) := \widetilde{\mathrm{Hol}}(K)/\sim$.

Definition 1.10.1 *Let A be an operator on $\mathcal{X}$ and $f \in \mathrm{Hol}(\mathrm{sp}^{\mathrm{ext}} A)$. Let γ be a contour in a domain of f that encircles $\mathrm{sp}^{\mathrm{ext}} A$ counterclockwise. We define*

$$f(A) := \frac{1}{2\pi \mathrm{i}} \int_\gamma (z - A)^{-1} f(z) \mathrm{d}z \qquad (1.10.7)$$

Clearly, the definition is independent of the choice of the contour.

Note that if $\mathrm{sp}A^{\mathrm{ext}}$ is the whole Riemann sphere (or equivalently $\mathrm{sp}A = \mathbb{C}$), then the functional calculus is trivial, since $\mathrm{Hol}(\mathbb{C}\cup\{\infty\})$ coincides with constant functions.

Theorem 1.10.2

$$\mathrm{Hol}(\mathrm{sp}^{\mathrm{ext}}A) \ni f \mapsto f(A) \in B(\mathcal{X}) \tag{1.10.8}$$

is a linear map satisfying

(1) $fg(A) = f(A)g(A)$*;*

(2) $1(A) = \mathbb{1}$*;*

(3) *If* $A \in B(\mathcal{X})$, *then* $\mathrm{id} \in \mathrm{Hol}(\mathrm{sp}^{\mathrm{ext}}A)$ *for* $\mathrm{id}(z) = z$ *and* $\mathrm{id}(A) = A$.

(4) *If* $f(z) := \sum_{n=0}^{\infty} f_n z^n$ *is an analytic function defined by a series absolutely convergent in a disk of radius greater than* $\mathrm{sr}A$, *then*

$$f(A) = \sum_{n=0}^{\infty} f_n A^n;$$

(5) *(Spectral mapping theorem).* $\mathrm{sp} f(A) = f(\mathrm{sp}^{\mathrm{ext}} A)$

(6) $g \in \mathrm{Hol}(f(\mathrm{sp}^{\mathrm{ext}} A)) \Rightarrow g \circ f(A) = g(f(A))$,

(7) $\|f(A)\| \le c_{\gamma,A} \sup_{z\in\gamma} |f(z)|$.

Proof. It is obvious that $1(A) = \mathbb{1}$. From the formula

$$(z - A)^{-1} = \sum_{n=0}^{\infty} z^{-n-1} A^n, \quad |z| > \mathrm{sr}(A),$$

we get that $\mathrm{id}(A) = A$.

It is clear that $f \to f(A)$ is linear. Let us show that it is multiplicative. Let $f_1, f_2 \in \mathrm{Hol}(\mathrm{sp} A)$. Choose a contour γ_2 around the contour γ_1, both in the domains of f_1 and f_2.

$$\begin{aligned}
&(2\pi \mathrm{i})^{-2} \int_{\gamma_1} f_1(z_1)(z_1 - A)^{-1} \mathrm{d}z_1 \int_{\gamma_2} f_2(z_2)(z_2 - A)^{-1} \mathrm{d}z_2 \\
&= (2\pi \mathrm{i})^{-2} \int_{\gamma_1} \int_{\gamma_2} f_1(z_1) f_2(z_2) \left((z_1 - A)^{-1} - (z_2 - A)^{-1}\right) (z_2 - z_1)^{-1} \mathrm{d}z_1 \mathrm{d}z_2 \\
&= (2\pi \mathrm{i})^{-2} \int_{\gamma_1} f_1(z_1)(z_1 - A)^{-1} \mathrm{d}z_1 \int_{\gamma_2} (z_2 - z_1)^{-1} f_2(z_2) \mathrm{d}z_2 \\
&+ (2\pi \mathrm{i})^{-2} \int_{\gamma_2} f_2(z_2)(z_2 - A)^{-1} \mathrm{d}z_2 \int_{\gamma_1} (z_1 - z_2)^{-1} f_1(z_1) \mathrm{d}z_1.
\end{aligned}$$

But

$$\int_{\gamma_1}(z_1 - z_2)^{-1} f_1(z_1) \mathrm{d}z_1 = 0,$$
$$\int_{\gamma_2}(z_2 - z_1)^{-1} f_2(z_2) \mathrm{d}z_2 = 2\pi \mathrm{i} f_2(z_1).$$

Thus

$$f_1(A) f_2(A) = f_1 f_2(A). \tag{1.10.9}$$

Let us prove the spectral mapping theorem. First we will show

$$\mathrm{sp} f(A) \subset f(\mathrm{sp}^{\mathrm{ext}} A). \tag{1.10.10}$$

If $\mu \not\in f(\mathrm{sp}^{\mathrm{ext}} A)$, then the function $z \mapsto f(z) - \mu \neq 0$ on $\mathrm{sp}^{\mathrm{ext}} A$. Therefore, $z \mapsto (f(z) - \mu)^{-1}$ belongs to $\mathrm{Hol}(\mathrm{sp}^{\mathrm{ext}} A)$. Thus $f(A) - \mu$ is invertible and therefore, $\mu \not\in \mathrm{sp} f(A)$. This implies (1.10.10).

Let us now show

$$\mathrm{sp} f(A) \supset f(\mathrm{sp}^{\mathrm{ext}} A). \tag{1.10.11}$$

Let $\mu \not\in \mathrm{sp} f(A)$. This clearly implies that $f(A) - \mu$ is invertible.

If μ does not belong to the image of f, then of course it does not belong to $f(\mathrm{sp}^{\mathrm{ext}}A)$. Let us assume that $\mu = f(\lambda)$. Then the function

$$z \mapsto g(z) := (f(z) - \mu)(z - \lambda)^{-1}$$

belongs to $\mathrm{Hol}(\mathrm{sp}^{\mathrm{ext}}A)$. Hence $g(A)$ is well defined as an element of $B(\mathcal{X})$. We check that $g(A)(f(A) - f(\lambda))^{-1} = (\lambda - A)^{-1}$. Hence $\lambda \not\in \mathrm{sp}^{\mathrm{ext}}A$. Thus $\mu \not\in f(\mathrm{sp}A)$. Consequently, (1.10.11) holds.

Let us show now (6). Notice that if $w \not\in f(\mathrm{sp}^{\mathrm{ext}}A)$, then the function $z \mapsto (w - f(z))^{-1}$ is analytic on a neighborhood of

$$(w - f(A))^{-1} = \frac{1}{2\pi i}\int_\gamma (w - f(z))^{-1}(z - A)^{-1}\mathrm{d}z.$$

We compute

$$
\begin{aligned}
&g(f(A))\\
&= \tfrac{1}{2\pi \mathrm{i}} \int_{\tilde\gamma} g(w)(w - f(A))^{-1} \mathrm{d}w\\
&= \tfrac{1}{(2\pi \mathrm{i})^2} \int_{\tilde\gamma} \int_{\gamma} g(w)(w - f(z))^{-1}(z - A)^{-1} \mathrm{d}w \mathrm{d}z\\
&= \tfrac{1}{(2\pi \mathrm{i})^2} \int_{\gamma} (z - A)^{-1} \mathrm{d}z \int_{\tilde\gamma} g(w)(w - f(z))^{-1} \mathrm{d}w\\
&= \tfrac{1}{2\pi \mathrm{i}} \int_{\gamma} g(f(z))(z - A)^{-1} \mathrm{d}z.
\end{aligned}
$$

□

1.11 Spectral idempotents

Let Ω be a subset of $B \subset \mathbb{C} \cup \{\infty\}$. Ω will be called an isolated subset of B, if $\Omega \cap (B \backslash \Omega)^{\mathrm{cl}} = \emptyset$ and $\Omega^{\mathrm{cl}} \cap (B \backslash \Omega) = \emptyset$ (or Ω is closed and open in the relative topology of B).

If B is in addition closed, then Ω is isolated iff both Ω and $(B \backslash \Omega)^{\mathrm{cl}}$ are closed in $\mathbb{C} \cup \{\infty\}$.

Let Ω be an isolated subset of $\mathrm{sp}^{\mathrm{ext}} A$. It is easy to see that we can find open non-intersecting neighbohoods of Ω and $\mathrm{sp}^{\mathrm{ext}} A \backslash \Omega$. Hence

$$\mathbb{1}_\Omega(z) := \begin{cases} 1 & z \text{ belongs to a neighborhood of } \Omega, \\ 0 & z \text{ belongs to a neighborhood of } \mathrm{sp}^{\mathrm{ext}} A \backslash \Omega. \end{cases}$$

defines an element of $\mathrm{Hol}(\mathrm{sp}^{\mathrm{ext}} A)$.

Clearly, $\mathbb{1}_\Omega^2 = \mathbb{1}_\Omega$. Hence $\mathbb{1}_\Omega(A)$ is an idempotent.

If γ is a counterclockwise contour around Ω outside of $\mathrm{sp}^{\mathrm{ext}} A \backslash \Omega$ then

$$\mathbb{1}_\Omega(A) = \frac{1}{2\pi \mathrm{i}} \int_\gamma (z - A)^{-1} \mathrm{d}z$$

This operator will be called the spectral idempotent of the operator A onto Ω.

$$\mathrm{sp}^{\mathrm{ext}} \Big(A\big|_{\mathrm{Ran}\, 1_\Omega(A)} \Big) = \mathrm{sp}^{\mathrm{ext}} A \cap \Omega.$$

If Ω_1 and Ω_2 are two isolated subsets of $\mathrm{sp}^{\mathrm{ext}} A$, then

$$\mathbb{1}_{\Omega_1}(A) \mathbb{1}_{\Omega_2}(A) = \mathbb{1}_{\Omega_1 \cap \Omega_2}(A)$$

1.12 Examples of unbounded operators

Example 1.12.1 *Let I be an infinite set and $(a_i)_{i\in I}$ be an unbounded complex sequence. Let $C_0(I)$ be the space of sequences with a finite number of non-zero elements. For $1 \le p < \infty$ we define the operator*

$$L^p(I) \supset C_0(I) \ni x \mapsto Ax \in L^p(I)$$

by the formula

$$(Ax)_i = a_i x_i.$$

(We can use $C_\infty(I)$ instead of $L^p(I)$, then $p = \infty$ in the formulas below). Then the operator A is unbounded and non-closed. Besides,

$$\mathrm{sp_p}(A) = \{a_i \ : \ i \in I\},$$

$$\mathrm{sp}A = \mathbb{C}.$$

The closure of A has the domain

$$\mathrm{Dom}\, A^{\mathrm{cl}} := \{(x_i)_{i\in I} \in L^p(I) \ : \ \sum_{i\in I} |a_i x_i|^p < \infty\} \tag{1.12.12}$$

We then have

$$\mathrm{sp}_{\mathrm{p}}(A^{\mathrm{cl}}) = \{a_i \ : \ i \in I\},$$
$$\mathrm{sp} A^{\mathrm{cl}} = \{a_i \ : \ i \in I\}^{\mathrm{cl}}.$$

To prove this let $\mathcal{D}$ be the rhs of (1.12.12) and $x \in \mathcal{D}$. Then there exists a countable set I_1 such that $i \notin I_1$ implies $x_i = 0$. We enumerate the elements of I_1: $i_1, i_2, \dots$. Define $x^n \in C_0(I)$ setting $x^n_{i_j} = x_{i_j}$ for $j \leq n$ and $x^n_i = 0$ for the remaining indices. Then $\lim_{n\to\infty} x^n = x$ and $Ax^n \to Ax$. Hence, $\{(x, Ax) \ : \ x \in \mathcal{D}\} \subset (\mathrm{Gr}\, A)^{\mathrm{cl}}$.

If x^n belongs to (1.12.12) and $(x^n, Ax^n) \to (x, y)$, then $x^n_i \to x_i$ and $a_i x^n_i = (Ax^n)_i \to y_i$. Hence $y_i = a_i x_i$. Using that $y \in L^p(I)$ we see that x belongs to (1.12.12).

Example 1.12.2 *Let $p^{-1} + q^{-1} = 1$, $1 < p \leq \infty$ and let $(w_i)_{i\in I}$ be a sequence that does not* *belong to $L^q(I)$. Let $C_0(I)$ be as above. Define*

$$L^p(I) \supset C_0(I) \ni x \mapsto \langle w|x\rangle := \sum_{i\in I} x_i w_i \in \mathbb{C}.$$

Then $\langle w|$ is non-closable.

It is sufficient to assume that $I = \mathbb{N}$ and define $v_i^n := \frac{|w_i|^q}{w_i(\sum_{i=1}^n |w_i|^q)}$, $i \leq n$, $v_i^n = 0$, $i > n$. Then $\langle w|v^n\rangle = 1$ and $\|v^n\|_p = (\sum_{i=1}^n |w_i|^q)^{-\frac{1}{q}} \to 0$. Hence $(0,1)$ belongs to the closure of the graph of the operator.

1.13 Pseudoresolvents

Definition 1.13.1 *Let $\Omega \subset \mathbb{C}$ be open. Then the continuous function*

$$\Omega \ni z \mapsto R(z) \in B(\mathcal{X})$$

is called a pseudoresolvent if

$$R(z_1) - R(z_2) = (z_2 - z_1)R(z_1)R(z_2). \tag{1.13.13}$$

Evidently, if A is a closed operator and $\Omega \subset \mathrm{rs}A$, then $\Omega \ni z \mapsto (z-A)^{-1}$ is a pseudoresolvent.

Proposition 1.13.2 *Let $\Omega \ni z \mapsto R_n(z) \in B(\mathcal{X})$ be a sequence of pseudoresolvents and $R(z) := \mathrm{s}-\lim\limits_{n\to\infty} R_n(z)$. Then $R(z)$ is a pseudoresolvent.*

Theorem 1.13.3 *Let $\Omega \ni z \mapsto R(z) \in B(\mathcal{X})$ be a pseudoresolvent. Then*

(1) $\mathcal{R} := \mathrm{Ran}\, R(z)$ *does not depend on* $z \in \Omega$.

(2) $\mathcal{N} := \mathrm{Ker} R(z)$ *does not depend on* $z \in \Omega$.

(3) $R(z)$ *is an analytic function and*

$$\frac{\mathrm{d}}{\mathrm{d}z} R(z) = -R(z)^2.$$

(4) $R(z)$ *is a resolvent of a certain operator iff* $\mathcal{N} = \{0\}$. *The domain of this operator equals* $\mathcal{R}$.

Proof. Let us prove $(4)\Leftarrow$. Fix $z_1 \in \Omega$. If $\mathcal{N} = \{0\}$, then every element of $\mathcal{R}$ can be uniquely represented as $R(z_1)x$, $x \in \mathcal{X}$. Define $AR(z_1)x := -x + z_1 R(z_1)x$. By formula (1.13.13) we check that the definition of A does not depend on z_1. $\square$

Chapter 2

One-parameter semigroups on Banach spaces

2.1 (M, β)-type semigroups

Let $\mathcal{X}$ be a Banach space.

Definition 2.1.1 $[0, \infty[\ni t \mapsto W(t) \in B(\mathcal{X})$ *is called a strongly continuous one-parameter semigroup iff*

(1) $W(0) = \mathbb{1}$;

(2) $W(t_1)W(t_2) = W(t_1 + t_2)$, $t_1, t_2 \in [0, \infty[$;

(3) $\lim\limits_{t \searrow 0} W(t)x = x$, $x \in \mathcal{X}$;

(4) *for some* $t_0 > 0$, $\|W(t)\| < M$, $0 \leq t \leq t_0$.

Remark 2.1.2 *Using the Banach-Steinhaus Theorem one can show that (4) follows from the remaining points.*

Theorem 2.1.3 *Let* $W(t)$ *e a strongly continuous semigroup. Then*

(1) *Besides, there exist constants* M, β *such that*

$$\|W(t)\| \leq M\mathrm{e}^{\beta t}; \tag{2.1.1}$$

(2) $[0, \infty[\times X \ni (t, x) \mapsto W(t)x \in \mathcal{X}$ *is a continuous function.*

Proof. By (4), for $t \leq nt_0$ we have $\|W(t)\| \leq M^n$. Hence, $\|W(t)\| \leq M \exp(\frac{t}{t_0} \log M)$. Therefore, (2.1.1) is satisfied.

Let $t_n \to t$ and $x_n \to x$. Then

$$\begin{aligned}\|W(t_n)x_n - W(t)x\| &\leq \|W(t_n)x_n - W(t_n)x\| + \|W(t_n)x - W(t)x\| \\ &\leq M\mathrm{e}^{\beta t_n}\|x_n - x\| + M\mathrm{e}^{\beta\min(t_n,t)}\|W(|t-t_n|)x - x\|.\end{aligned}$$

□

We say that the semigroup $W(t)$ is (M,β)-type, if the condition (2.1.1) is satisfied.

Clearly, if $W(t)$ is (M,β)-type, then $W(t)\mathrm{e}^{-\beta t}$ is $(M,0)$-type. Since $W(0) = \mathbb{1}$, no semigroups (M,β) exist for $M < 1$.

2.2 Generator of a semigroup

If $W(t)$ is a strongly continuous one-parameter semigroup, we define

$$\mathrm{Dom}\, A := \{x \in \mathcal{X} \ : \ \text{there exists } \lim_{t \searrow 0} t^{-1}(W(t)x - x)\}.$$

The operator A with the domain $\operatorname{Dom} A$ is defined by the formula

$$Ax := \lim_{t \searrow 0} t^{-1}(W(t)x - x).$$

A will be called the generator of $W(t)$. In the following theorem we show that an operator cannot be the generator of more than one semigroup.

If $W(t)$ is the semigroup generated by A, then we will write $W(t) =: \mathrm{e}^{tA}$.

Theorem 2.2.1 (1) A *is a closed densely defined operator;*

(2) $W(t) \operatorname{Dom} A \subset \operatorname{Dom} A$ *and* $W(t)A = AW(t)$*;*

(3) *If* $W_1(t)$ *and* $W_2(t)$ *are two different semigroups, then their generators are different.*

Proof of Theorem 2.2.1 (2). Let $x \in \operatorname{Dom} A$. Then

$$\lim_{s \searrow 0} s^{-1}(W(s) - \mathbb{1})W(t)x = W(t) \lim_{s \searrow 0} s^{-1}(W(s) - \mathbb{1})x = W(t)Ax. \tag{2.2.2}$$

Hence the limit of the left hand side of (2.2.2) exists. Hence $W(t)x \in \operatorname{Dom} A$ and $AW(t)x = W(t)Ax$. $\square$

Lemma 2.2.2 *For $x \in \mathcal{X}$ put*

$$B_t x := t^{-1} \int_0^t W(s) x \mathrm{d}s.$$

Then

(1) $\mathrm{s-}\lim\limits_{t \searrow 0} B_t = \mathbb{1}$.

(2) $B_t W(s) = W(s) B_t$.

(3) *For* $x \in \mathrm{Dom}\, A$, $AB_t x = B_t A x$.

(4) *If* $x \in \mathcal{X}$, *then* $B_t x \in \mathrm{Dom}\, A$,

$$AB_t x = t^{-1}(W(t)x - x). \tag{2.2.3}$$

(5) *If* $\lim_{t \searrow 0} AB_t x$ *exists, then* $x \in \mathrm{Dom}\, A$ *and the limit equals* Ax.

Proof. (1) follows by

$$B_t x - x = t^{-1} \int_0^t (W(s)x - x) \mathrm{d}s \underset{t \searrow 0}{\to} 0.$$

(2) is obvious. (3) is proven as Theorem 2.2.1 (2). To prove (4) we note that

$$u^{-1}(W(u) - \mathbb{1})B_t x = t^{-1}(W(t) - \mathbb{1})B_u x \underset{u \searrow 0}{\rightarrow} t^{-1}(W(t)x - x).$$

(5) follows from (4). □

Proof of Theorem 2.2.1 (1) The density of $\mathrm{Dom}\, A$ follows by Lemma 2.2.2 (1) and (3).

Let us show that A is closed. Let $x_n \underset{n\to\infty}{\rightarrow} x$ and $Ax_n \underset{n\to\infty}{\rightarrow} y$. Using the boundedness of $B_t A = AB_t$ we get

$$B_t y = \lim_{n\to\infty} B_t A x_n = \lim_{n\to\infty} AB_t x_n = AB_t x.$$

Hence

$$y = \lim_{t\downarrow 0} B_t y = \lim_{t\downarrow 0} AB_t x. \tag{2.2.4}$$

By Lemma 2.2.2 (5), $x \in \mathrm{Dom}\, A$ and (2.2.4) equals Ax. □

Proposition 2.2.3 *Let $W(t)$ be a semigroup and A its generator. Then, for any $x \in \mathrm{Dom}\, A$*

there exists a unique solution of

$$[0,\infty[\ni t \mapsto x(t) \in \mathrm{Dom}\, A, \quad \frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t), \quad x(0) = x. \tag{2.2.5}$$

(for $t = 0$ the derivative is right-sided). The solution is given by $x(t) = W(t)x$.

Proof. Let us show that $x(t) := W(t)x$ solves (2.2.5), both for the left and right derivative:

$$\begin{aligned} u^{-1}(W(t+u)x - W(t)x) &= W(t)u^{-1}(W(u)-1)x \underset{u\downarrow 0}{\to} W(t)Ax = AW(t)x, \\ u^{-1}(W(t-u)x - W(t)x) &= W(t-u)u^{-1}(W(u)-1)x \underset{u\downarrow 0}{\to} W(t)Ax = AW(t)x, \quad 0 \le u \le t. \end{aligned}$$

Let us show now the uniqueness. Let $x(t)$ solve (2.2.5). Let $y(s) := W(t-s)x(s)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}y(s) = W(t-s)Ax(s) - AW(t-s)x(s) = 0$$

Hence $y(s)$ does not depend on s. At $s = t$ it equals $x(t)$, and at $s = 0$ it equals $W(t)x$. □

Proof of Theorem 2.2.1 (3) By Prop. 2.2.3 (2), $W(t)$ is uniquely determined by A on $\mathrm{Dom}\, A$. But $W(t)$ is bounded and $\mathrm{Dom}\, A$ is dense, hence $W(t)$ is uniquely determined. □

2.3 One-parameter groups

Definition 2.3.1 $\mathbb{R} \ni t \mapsto W(t) \in B(\mathcal{X})$ *is called a strongly continuous one-parameter group iff*

(1) $W(0) = \mathbb{1}$*;*

(2) $W(t_1)W(t_2) = W(t_1 + t_2)$, $t_1, t_2 \in \mathbb{R}$*;*

(3) $\lim\limits_{t \to 0} W(t)x = x$, $x \in \mathcal{X}$*;*

(4) *for some* $t_0 > 0$, $\|W(t)\| < M$, $|t| \leq t_0$.

Each 1-parameter group $\mathbb{R} \ni t \mapsto W(t)$ consists of two semigroups:

$$[0, \infty[\ni t \mapsto W(t), \quad [0, \infty[\ni t \mapsto W(-t).$$

If A is the generator of the former, then $-A$ is the generator of the latter.

Conversely, if both A and $-A$ generate semigroups, then they can be combined to form a group.

2.4 Norm continuous semigroups

Theorem 2.4.1 (1) *If $A \in B(\mathcal{X})$, then $\mathbb{R} \ni z \mapsto \mathrm{e}^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$ is a norm continuous group and A is its generator.*

(2) *If a one-parameter semigroup $W(t)$ is norm continuous, then its generator is bounded.*

Proof. (1) follows by the functional calculus.

Let us show (2). $W(t)$ is norm continuous, hence $\lim_{t \to 0} B_t = 1\!\!1$. Therefore, for $0 < t < t_0$

$$\|B_t - 1\!\!1\| < 1.$$

Hence B_t is then invertible.

We know that for $x \in \mathrm{Dom}\, A$

$$t^{-1}(W(t) - 1\!\!1)x = B_t A x.$$

For $0 \leq t < t_0$ we can write this as

$$Ax = t^{-1} B_t^{-1} (W(t) - 1) x.$$

Hence $\|Ax\| \leq c\|x\|$. $\square$

2.5 Essential domains of generators

Theorem 2.5.1 *Let $W(t)$ be a strongly continuous one-parameter semigroup and let A be its generator. Let $\mathcal{D} \subset \mathrm{Dom}\, A$ be dense in $\mathcal{X}$ and $W(t)\mathcal{D} \subset \mathcal{D}$, $t > 0$. Then $\mathcal{D}$ is dense in $\mathrm{Dom}\, A$ in the graph topology—in other words, $\mathcal{D}$ is an essential domain of A.*

Lemma 2.5.2 (1) *For $x \in \mathcal{X}$, $\|B_t x\|_{\mathrm{Dom}\, A} \leq (Ct^{-1} + 1)\|x\|$;*

(2) *For $x \in \mathrm{Dom}\, A$, $\lim_{t \downarrow 0} \|B_t x - x\|_{\mathrm{Dom}\, A} = 0$;*

(3) *$W(t)$ is a strongly continuous semi-group on $\mathrm{Dom}\, A$ equipped with the graph norm.*

(4) *If $\tilde{\mathcal{D}}$ is a closed subspace in $\mathrm{Dom}\, A$ invariant wrt $W(t)$, then it is invariant also wrt B_t.*

Proof. (1) follows by Lemma 2.2.2 (3).

(2) follows by Lemma 2.2.2 (1) and because $B(t)$ commutes with A.

(3) follows from the fact that $W(t)$ is a strongly continuous semigroup on $\mathcal{X}$, preserves $\operatorname{Dom} A$ and commutes with A.

To show (4), note that $B_t x$ is defined using an integral involving $W(s)x$. $W(s)x$ depends continuously on s in the topology of $\operatorname{Dom} A$, as follows by (3). Hence this integral (as Riemann's integral) is well defined. Besides, $B_t x$ belongs to the closure of the space spanned by $W(s)x$, $0 \le s \le t$. $\square$

Proof of Theorem 2.5.1. Let $x \in \operatorname{Dom} A$, $x_n \in \mathcal{D}$ and $x_n \underset{n\to\infty}{\to} x$ in $\mathcal{X}$. Let $\tilde{\mathcal{D}}$ be he closure of $\mathcal{D}$ in $\operatorname{Dom} A$. Then $B_t x_n \in \tilde{\mathcal{D}}$, by Lemma 2.5.2 (4). By Lemma 2.5.2 (1) we have

$$\|B_t x_n - B_t x\|_{\operatorname{Dom} A} \le C_t \|x_n - x\|.$$

Hence $B_t x \in \tilde{\mathcal{D}}$. By Lemma 2.5.2 (2)

$$\|B_t x - x\|_{\operatorname{Dom} A} \underset{t\downarrow 0}{\to} 0.$$

Hence, $x \in \tilde{\mathcal{D}}$. $\square$

2.6 Operators of (M, β)-type

Theorem 2.6.1 *Let A be a densely defned operator. Then the following conditions are equivalent:*

(1) $[\beta, \infty[\subset \mathrm{rs}(A)$ *and*

$$\|(x - A)^{-m}\| \leq M|x - \beta|^{-m}, \quad m = 1, 2, \ldots, \quad x \in \mathbb{R}, \ x > \beta$$

(2) $\{z \in \mathbb{C} \ : \ \mathrm{Re} z > \beta\} \subset \mathrm{rs}(A)$ *and*

$$\|(z - A)^{-m}\| \leq M|\mathrm{Re} z - \beta|^{-m}, \quad m = 1, 2, \ldots, \quad z \in \mathbb{C}, \ \mathrm{Re} z > \beta.$$

Proof. It suffices to prove (1)⇒(2). Let (1) be satisfied. It suffices to assume that $\beta = 0$.

Let $z = x + \mathrm{i}y$. Then for $t > 0$

$$\begin{aligned}(z-A)^{-m} &= (x+t-A)^m(\mathbb{1} + (\mathrm{i}y-t)(x+t-A)^{-1})^{-m} \\ &= \sum_{j=0}^{\infty}(x+t-A)^{-m-j}(\mathrm{i}y-t)^j \begin{pmatrix} -m \\ j \end{pmatrix}.\end{aligned}$$

Using the fact that $\left|\begin{pmatrix} -m \\ j \end{pmatrix}\right| = (-1)^j \begin{pmatrix} -m \\ j \end{pmatrix}$ we get

$$\begin{aligned}\|(z-A)^{-m}\| &\leq M\sum_{j=0}^{\infty}|x+t|^{-m-j}(-1)^j|\mathrm{i}y-t|^j \begin{pmatrix} -m \\ j \end{pmatrix} \\ &= M|x+t|^m\left(1 - \tfrac{|\mathrm{i}y-t|}{x+t}\right)^{-m} \\ &= M(x+t-|\mathrm{i}y-t|)^{-m} \underset{t\to\infty}{\longrightarrow} Mx^{-m}.\end{aligned}$$

□

Definition 2.6.2 *We say that an operator A is (M,β)-type, iff the conditions of Theorem*

2.6.1 are satisfied.

Obviously, if A is of (M, β)-type, then $A - \beta$ is of $(M, 0)$-type.

2.7 The Hille-Philips-Yosida theorem

Theorem 2.7.1 *If $W(t)$ is a semigroup of (M, β)-type, then its generator A is also of (M, β)-type. Besides,*

$$(z - A)^{-1} = \int_0^\infty \mathrm{e}^{-tz} W(t) \mathrm{d}t, \quad \mathrm{Re} z > \beta.$$

Proof. Set

$$R(z)x := \int_0^\infty \mathrm{e}^{-zt} W(t) x \mathrm{d}t.$$

Let $y = R(z)x$. Then

$$\begin{aligned}
&u^{-1}(W(u) - \mathbb{1})y \\
= \;& -u^{-1}\mathrm{e}^{zu}\int_0^u \mathrm{e}^{-zt}W(t)x\mathrm{d}t + u^{-1}(\mathrm{e}^{zu} - 1)\int_0^\infty \mathrm{e}^{-zt}W(t)x\mathrm{d}t \underset{u\searrow 0}{\to} -x + zy.
\end{aligned}$$

Hence $y \in \operatorname{Dom} A$ and $(z - A)R(z)x = x$.

Suppose now that $x \in \operatorname{Ker}(z - A)$. Then $x_t := \mathrm{e}^{zt}x \in \operatorname{Dom} A$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}x_t = Ax_t$. Hence $x_t = W(t)x$. But $\|x_t\| = \mathrm{e}^{\mathrm{Re}zt}\|x\|$, which is impossible.

By the formula

$$(z - A)^{-m} = \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-z(t_1+\cdots+t_m)}W(t_1 + \cdots + t_m)\mathrm{d}t_1 \cdots \mathrm{d}t_m$$

we get the estimate

$$\|(z - A)^{-m}\| \leq \int_0^\infty \cdots \int_0^\infty M\mathrm{e}^{-(z-\beta)(t_1+\cdots+t_m)}\mathrm{d}t_1 \cdots \mathrm{d}t_m = M|z - \beta|^{-m}.$$

$\square$

Theorem 2.7.2 *If A is an operator of (M,β)-type, then it is the generator of a semigroup of (M,β)-type.*

To simplify, let us assume that $\beta = 0$ (which does not restrict the generality). Then we have the formula

$$\mathrm{e}^{tA} = \mathrm{s-}\lim_{n\to\infty}\left(1\!\!1 - \frac{t}{n}A\right)^{-n},$$

$$\left\|\mathrm{e}^{tA}x - \left(1\!\!1 - \frac{t}{n}A\right)^{-n}x\right\| \leq M\frac{t^2}{2}\|A^2x\|,\quad x \in \mathrm{Dom}\, A^2.$$

Proof. Set

$$V_n(t) := \left(1\!\!1 - \frac{t}{n}A\right)^{-n}.$$

Let us first show that

$$\mathrm{s-}\lim_{t\downarrow 0} V_n(t) = 1\!\!1. \tag{2.7.6}$$

To prove (2.7.6) it suffices to prove that

$$\mathrm{s-}\lim_{s\downarrow 0}(\mathbb{1} - sA)^{-1} = \mathbb{1}. \tag{2.7.7}$$

We have $(\mathbb{1} - sA)^{-1} - \mathbb{1} = (s^{-1} - A)^{-1}A$. Hence for $x \in \mathrm{Dom}\, A$

$$\|(\mathbb{1} - sA)^{-1}x - x\| \le Ms^{-1}\|Ax\|,$$

which proves (2.7.7).

Let us list some other properties of $V_n(t)$: for $\mathrm{Re}t > 0$, $V_n(t)$ is holomorphic, $\|V_n(t)\| \le M$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}V_n(t) = A\left(\mathbb{1} - \frac{t}{n}A\right)^{-n-1}.$$

To show that $V_n(t)x$ is a Cauchy sequence for $x \in \mathrm{Dom}(A^2)$, we compute

$$\begin{aligned} V_n(t)x - V_m(t)x &= \lim_{s\downarrow 0} V_n(t-s)V_m(s)x - \lim_{s\uparrow t} V_n(t-s)V_m(s)x \\ &= \lim_{\epsilon\downarrow 0} \int_\epsilon^{t-\epsilon} \frac{\mathrm{d}}{\mathrm{d}s} V_n(t-s)V_m(s)x \\ &= \lim_{\epsilon\downarrow 0} \int_\epsilon^{t-\epsilon} \Big(- V_n'(t-s)V_m(s) + V_n(t-s)V_m'(s)\Big)x \\ &= \lim_{\epsilon\downarrow 0} \int_\epsilon^{t-\epsilon} \left(\tfrac{s}{n} - \tfrac{t-s}{m}\right) \left(\mathbb{1} - \tfrac{t-s}{n}A\right)^{-n-1} \left(\mathbb{1} - \tfrac{s}{n}A\right)^{-m-1} A^2 x. \end{aligned}$$

Hence for $x \in \mathrm{Dom}(A^2)$

$$\begin{aligned} \|V_n(t)x - V_m(t)x\| &\le \|A^2 x\| \int_0^t |\tfrac{s}{m} - \tfrac{t-s}{n}| M^2 \mathrm{d}s \\ &= M^2 (\tfrac{1}{n} + \tfrac{1}{m}) \tfrac{t^2}{2}. \end{aligned}$$

By the Proposition 1.9.2, $\mathrm{Dom}(A^2)$ is dense in $\mathcal{X}$. Therefore, there exists a limit uniform on $[0, t_0]$

$$\mathrm{s}-\lim_{n\to\infty} V_n(t) =: W(t),$$

which depends strongly continuously on t.

Finally, let us show that $W(t)$ is a semigroup with the generator A. To this end it suffices to show that for $x \in \mathrm{Dom}\, A$

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t)x = AW(t)x. \tag{2.7.8}$$

But $x \in \mathrm{Dom}\, A$

$$V_n(t+u)x = V_n(t)x + \int_t^{t+u} A\Big(\mathbb{1} - \frac{s}{n}A\Big)^{-1} V_n(s)x\mathrm{d}s$$

Hence passing to the limit we get

$$W(t+u)x = W(t)x + \int_t^{t+u} AW(s)x\mathrm{d}s.$$

This implies (2.7.8). □

2.8 Semigroups of contractions and their generators

Theorem 2.8.1 *Let A be a closed operator on $\mathcal{X}$. Then the following conditions are eqivalent:*

(1) A *is a generator of a semigroup of contractions, i.e.* $\|e^{tA}\| \leq 1$, $t \geq 0$.

(2) *The operator* A *is of* $(1,0)$*-type.*

(3) $]0,\infty[\subset \mathrm{rs}(A)$ *and*

$$\|(\mu - A)^{-1}\| \leq \mu^{-1}, \quad \mu \in \mathbb{R},\ \mu > 0,$$

(4) $\{z \in \mathbb{C}\ :\ \mathrm{Re} z > 0\} \subset \mathrm{rs}(A)$ *and*

$$\|(z - A)^{-1}\| \leq |\mathrm{Re} z|^{-1}, \quad z \in \mathbb{C},\ \mathrm{Re} z > 0.$$

Proof. The equivalence of (1) and (2) is a special case of Theorems 2.7.1 and 2.7.2. The implications (2)⇒(3) and (2)⇒(4) are obvious, the converse implications are easy. □

Chapter 3

Unbounded operators on Hilbert spaces

3.1 Graph scalar product

Let $\mathcal{V}$, $\mathcal{W}$ be Hilbert spaces. Let $A : \mathcal{V} \to \mathcal{W}$ be an operator with domain $\mathrm{Dom}\, A$. It is natural to treat $\mathrm{Dom}\, A$ as a space with the graph scalar product

$$(v_1|v_2)_A := (v_1|v_2) + (Av_1|Av_2).$$

Clearly, $\mathrm{Dom}\, A$ is a Hilbert space with the graph scalar product iff A is closed.

3.2 The adjoint of an operator

Definition 3.2.1 *Let $A : \mathcal{V} \to \mathcal{W}$ have a dense domain. Then $w \in \mathrm{Dom}\, A^*$, iff the functional*

$$\mathrm{Dom}\, A \ni v \mapsto (w|Av)$$

is bounded (in the topology of $\mathcal{V}$). Hence there exists a unique $y \in \mathcal{V}$ such that

$$(w|Av) = (y|v), \quad v \in \mathcal{V}.$$

The adjoint of A is then defined by setting

$$A^* w = y.$$

Theorem 3.2.2 *Let $A : \mathcal{V} \to \mathcal{W}$ have a dense domain. Then*

(1) A^* *is closed.*

(2) $\mathrm{Dom}\, A^*$ *is dense in $\mathcal{W}$ iff A is closable.*

(3) $(\mathrm{Ran}\, A)^\perp = \mathrm{Ker} A^*$.

(4) $\mathrm{Dom}\, A \cap (\mathrm{Ran}\, A^*)^\perp \supset \mathrm{Ker} A$.

Proof. Let $j : \mathcal{V} \oplus \mathcal{W} \to \mathcal{W} \oplus \mathcal{V}$, $j(v, w) := (-w, v)$. Note that j is unitary. We have

$$\mathrm{Gr}\, A^* = j(\mathrm{Gr}\, A)^\perp.$$

Hence $\mathrm{Gr}\, A^*$ is closed. This proves (1).

Let us prove (2).

$$\begin{aligned} w \in (\mathrm{Dom}\, A^*)^\perp \ &\Leftrightarrow (0, w) \in (\mathrm{Gr}\, A^*)^\perp = j(\mathrm{Gr}\, A)^{\perp\perp} \\ &\Leftrightarrow (w, 0) \in (\mathrm{Gr}\, A)^{\perp\perp} = (\mathrm{Gr}\, A)^{\mathrm{cl}}. \end{aligned}$$

Proof of (3):

$$
\begin{aligned}
w \in \mathrm{Ker} A^* &\Leftrightarrow (A^* w|v) = 0, \quad v \in \mathcal{V} \\
&\Leftrightarrow (A^* w|v) = 0, \quad v \in \mathrm{Dom}\, A \\
&\Leftrightarrow (w|Av) = 0, \quad v \in \mathrm{Dom}\, A \\
&\Leftrightarrow w \in (\mathrm{Ran}\, A)^{\perp}.
\end{aligned}
$$

Proof of (4)

$$
\begin{aligned}
v \in \mathrm{Ker} A &\Leftrightarrow (w|Av) = 0, \quad w \in \mathcal{W} \\
&\Rightarrow (w|Av) = 0, \quad w \in \mathrm{Dom}\, A^* \\
&\Leftrightarrow (A^* w|v) = 0, \quad w \in \mathrm{Dom}\, A^* \\
&\Leftrightarrow v \in (\mathrm{Ran}\, A^*)^{\perp}.
\end{aligned}
$$

Theorem 3.2.3 *Let $A : \mathcal{V} \to \mathcal{W}$ be closable with a dense domain. Then*

(1) A^* *is closed with a dense domain.*

(2) $A^* = (A^{\mathrm{cl}})^*$.

(3) $(A^*)^* = A^{\mathrm{cl}}$

(4) $(\mathrm{Ran}\, A)^\perp = \mathrm{Ker} A^*$. *Hence A^* is injective iff $\mathrm{Ran}\, A$ is dense.*

(5) $(\mathrm{Ran}\, A^*)^\perp = \mathrm{Ker} A$. *Hence A is injective iff $\mathrm{Ran}\, A^*$ is dense.*

Proof. (1) was proven in Theorem 3.2.2.

To see (2) note that

$$\mathrm{Gr}\, A^* = j(\mathrm{Gr}\, A)^\perp = j((\mathrm{Gr}\, A)^{\mathrm{cl}})^\perp = \mathrm{Gr}\, A^{\mathrm{cl}*}.$$

To see (3) we use

$$\mathrm{Gr}\ (A^{**}) = j^{-1}\left(j(\mathrm{Gr}\, A)^\perp\right)^\perp = (\mathrm{Gr}\, A)^{\perp\perp} = (\mathrm{Gr}\, A)^{\mathrm{cl}}.$$

(4) is proven in Theorem 3.2.2.

To prove (5) note that in the second line of the proof of Theorem 3.2.2 (4) we can use the fact that $\mathrm{Dom}\, A^*$ is dense in $\mathcal{W}$ to replace $\Rightarrow$ with $\Leftrightarrow$. $\square$

3.3 Inverse of the adjoint operator

Theorem 3.3.1 *Let A be densely defined, closed, injective and with a dense range. Then*

(1) A^{-1} *is densely defined, closed, injective and with a dense range.*

(2) A^* *is densely defined, closed, injective and with a dense range.*

(3) $(A^*)^{-1} = (A^{-1})^*$.

Proof. (1) and (2) sum up previously proven facts.

To prove (3), recall the maps $\tau, j : \mathcal{V} \oplus \mathcal{W} \to \mathcal{W} \oplus \mathcal{V}$. We have

$$\mathrm{Gr}\, A^* = j(\mathrm{Gr}\, A)^\perp, \quad \mathrm{Gr}\, A^{-1} = \tau(\mathrm{Gr}\, A).$$

Hence

$$\mathrm{Gr}\, A^{-1*} = j(\tau(\mathrm{Gr}\, A))^\perp = \tau^{-1}(j(\mathrm{Gr}\, A)^\perp) = \mathrm{Gr}\, A^{*-1}.$$

□

Theorem 3.3.2 *Let $A : \mathcal{V} \to \mathcal{W}$ be densely defined and closed. Then the following conditions* *are equivalent:*

(1) A *is invertible.*

(2) A^* *is invertible.*

(3) *For some* $c > 0$, $\|Av\| \geq c\|v\|$, $v \in \mathcal{V}$ *and* $\|A^*w\| \geq c\|v\|$, $w \in \mathcal{W}$.

Proof. (1)⇒(2). Let A be invertible. Then $A^{-1} \in B(\mathcal{W}, \mathcal{V})$. Hence, $A^{-1*} \in B(\mathcal{V}, \mathcal{W})$.

Clearly, the assumptions of Theorem 3.3.1 are satisfied, and hence $A^{*-1} = A^{-1*}$. Therefore, $A^{*-1} \in B(\mathcal{V}, \mathcal{W})$.

(1)⇐(2). A^* is also densely defined and closed. Hence the same arguments as above apply.

It is obvious that (1) and (2) imply (3). Let us prove that (3)⇒(1). $\|A^*v\| \geq c\|v\|$ implies that $\mathrm{Ker} A^* = \{0\}$. Hence $(\mathrm{Ran}\, A)^{\perp}$ is dense. This together with $\|Av\| \geq c\|v\|$ implies that $\mathrm{Ran}\, A = \mathcal{W}$, and consequently, A is invertible. □

Theorem 3.3.3 *Let* $A : \mathcal{V} \to \mathcal{W}$ *be densely defined and closed. Then* $\mathrm{sp}^{\mathrm{ext}}(A) = \overline{\mathrm{sp}^{\mathrm{ext}}(A^*)}$.

3.4 Numerical range and maximal operators

Definition 3.4.1 *Let T be an operator on $\mathcal{V}$. The numerical range of T is defined as*

$$\operatorname{Num} T := \{(v|Tv) \in \mathbb{C} \ : \ v \in \mathcal{V}, \ \|v\| = 1\}.$$

Theorem 3.4.2 (1) *In a two-dimensional space the numerical range is always an elipse together with its interior.*

(2) $\operatorname{Num} T$ *is a convex set.*

(3) $\operatorname{Num}(\alpha T + \beta \mathbb{1}) = \alpha \operatorname{Num}(T) + \beta$.

(4) $\operatorname{Num} T^* = \overline{\operatorname{Num} T}$.

(5) $\operatorname{Num}(T + S) \subset \operatorname{Num} T + \operatorname{Num} S$.

Proof. (1) We write $T = T_{\mathrm{R}} + \mathrm{i} T_{\mathrm{I}}$, where T_{R}, T_{I} are self-adjoint. We diagonalize T_{I}. Thus if $\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$ is the matrix of T, then $t_{12} = \bar{t}_{21}$. By multiplying one of the basis vectors with a phase factor we can guarantee that $t_{12} = t_{21}$ is real.

Now T is given by a matrix of the form

$$c\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda & \mu \\ \mu & -\lambda \end{bmatrix} + \mathrm{i}\begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}$$

Any normalized vector up to a phase factor equals $v = (\cos\alpha, \mathrm{e}^{\mathrm{i}\phi}\sin\alpha)$ and

$$(v|Tv) - c = \lambda\cos 2\alpha + \mu\cos\phi\sin 2\alpha + \mathrm{i}\gamma\cos 2\alpha =': x + \mathrm{i}y. \tag{3.4.1}$$

Now it is an elementary exercise to check that $x + \mathrm{i}y$ are given by (3.4.1), iff they satisfy

$$(\gamma x - \lambda y)^2 + \mu^2 y^2 \le \gamma^2\mu^2.$$

(2) follows immediately from (1). $\square$

Theorem 3.4.3 (1) $\|(z-T)v\| \ge \mathrm{dist}(z, \mathrm{Num}T)\|v\|, \quad v \in \mathrm{Dom}\, T.$

(2) *If T is a closed operator and $z \in \mathbb{C}\backslash(\mathrm{Num}T)^{\mathrm{cl}}$, then $z - T$ has a closed range.*

(3) *If $z \in \mathrm{rs}T\backslash\mathrm{Num}T$, then $\|(z-T)^{-1}\| \le |\mathrm{dist}(z, \mathrm{Num}T)|^{-1}$.*

(4) *Let Δ be a connected component of $\mathbb{C}\backslash(\mathrm{Num}T)^{\mathrm{cl}}$. Then either $\Delta \subset \mathrm{sp}T$ or $\Delta \subset \mathrm{rs}T$.*

Proof. To prove (1), take $z \notin (\mathrm{Num}T)^{\mathrm{cl}}$. Recall that $\mathrm{Num}T$ is convex. Hence, replacing T wih $\alpha T + \beta$ we can assume that $z = \mathrm{i}\nu$ and $0 \in (\mathrm{Num}T)^{\mathrm{cl}} \subset \{\mathrm{Im}z \leq 0\}$. Thus $\nu = \mathrm{dist}(\mathrm{i}\nu, \mathrm{Num}T)$ and

$$\begin{aligned}
\|(\mathrm{i}\nu - T)v\|^2 &= (Tv|Tv) - i\nu(v|Tv) + \mathrm{i}\nu(Tv|v) + |\nu|^2\|v\|^2 \\
&= (Tv|Tv) - 2\nu\mathrm{Im}(v|Tv) + |\nu|^2\|v\|^2 \\
&\geq |\nu|^2\|v\|^2.
\end{aligned}$$

(1) implies (2) and (3).

Let $z_0 \in \mathrm{rs}T \backslash \mathrm{Num}T$. By (3), if $r = \mathrm{dist}(z_0, \mathrm{Num}T)$, then $\{|z - z_0| < r\} \subset \mathrm{rs}T$. This proves (4). $\square$

Definition 3.4.4 *An operator T is called maximal, if $\mathrm{sp}T \subset (\mathrm{Num}T)^{\mathrm{cl}}$.*

Clearly, if T is a maximal operator, and $z \notin (\mathrm{Num}T)^{\mathrm{cl}}$, then

$$\|(z - T)^{-1}\| \leq (\mathrm{dist}(z, \mathrm{Num}T))^{-1}.$$

If T is bounded, then T is maximal.

Theorem 3.4.5 *Suppose that T is an operator and for any connected component Δ_i of $\mathbb{C}\backslash(\mathrm{Num}T)^{\mathrm{cl}}$ we choose $\lambda_i \in \Delta_i$. Then the following conditions are necessary and sufficient for T to be maximal*

(1) *For all i, $\lambda_i \notin \mathrm{sp}T$;*

(2) *T is closable and for all i, $\mathrm{Ran}\,(\lambda_i - T) = \mathcal{V}$.*

(3) *T is closed and for all i, $\mathrm{Ran}\,(\lambda_i - T)$ is dense in $\mathcal{V}$.*

(4) *T is closed and for all i, $\mathrm{Ker}(\overline{\lambda}_i - T^*) = \{0\}$.*

If K is a closed convex subset of $\mathbb{C}$, then $\mathbb{C}\backslash K$ is either connected or has two connected components.

3.5 Dissipative operators

Definition 3.5.1 *We say that an operator A is dissipative iff*

$$\mathrm{Im}(v|Av) \leq 0, \quad v \in \mathrm{Dom}\, A.$$

Equivalently, A is dissipative iff $\mathrm{Num}A \subset \{\mathrm{Im}z \leq 0\}$.

Definition 3.5.2 *A is maximally dissipative iff A is dissipative and $\mathrm{sp}A \subset \{\mathrm{Im}z \leq 0\}$.*

Theorem 3.5.3 *Let A be a densely defined operator. Then the following conditions are equivalent:*

(1) *$-\mathrm{i}A$ is the generator of a strongly continuous semigroup of contractions.*

(2) *A is maximally dissipative.*

Proof. (1) $\Rightarrow$(2) We have

$$\mathrm{Re}(v|\mathrm{e}^{-\mathrm{i}tA}v) \leq |(v|\mathrm{e}^{-\mathrm{i}tA}v)| \leq \|v\|^2.$$

Hence

$$\begin{aligned}\mathrm{Im}(v|Av) &= \mathrm{Re}(v|-\mathrm{i}Av)\\ &= \mathrm{Re}\lim_{t\nearrow 0} t^{-1}\left((v|\mathrm{e}^{-\mathrm{i}tA}v) - \|v\|^2\right) \leq 0.\end{aligned}$$

Hence A is dissipative.

We know that the generators of contractions satisfy $\{\mathrm{Re}z > 0\} \subset \mathrm{rs}(-\mathrm{i}A)$.

(2)⇒(1) Let $\mathrm{Re}z > 0$. We have

$$\begin{aligned}\|v\|\|(z+\mathrm{i}A)v\| &\geq |(v|(z+\mathrm{i}A)v)|\\ &\geq \mathrm{Re}(v|(z+\mathrm{i}A)v) \geq \mathrm{Re}z\|v\|^2.\end{aligned}$$

Hence, noting that $z \in \mathrm{rs}(-\mathrm{i}A)$, we obtain $\|(z+\mathrm{i}A)^{-1}\| \leq \mathrm{Re}z^{-1}$. Therefore, $-\mathrm{i}A$ is an operator of the type $(1,0)$. □

Theorem 3.5.4 *Let A be dissipative. Then the following conditions are equivalent:*

(1) *A is maximally dissipative.*

(2) *A is closable and there exists z_0 with $\mathrm{Im}z_0 > 0$ and $\mathrm{Ran}\,(z_0 - A) = \mathcal{V}$.*

(3) A *is closed and there exists* z_0 *with* $\mathrm{Im} z_0 > 0$ *and* $\mathrm{Ran}\,(z_0 - A)$ *dense in* $\mathcal{V}$.

(4) A *is closed and there exists* z_0 *with* $\mathrm{Im} z_0 > 0$ *and* $\mathrm{Ker}(\overline{z}_0 - A^*) = \{0\}$.

3.6 Hermitian operators

Definition 3.6.1 *An operator* $A : \mathcal{V} \to \mathcal{V}$ *is hermitian iff*

$$(Aw|v) = (w|Av), \;\; w, v \in \mathrm{Dom}\, A.$$

Equivalently, A is hermitian iff $\mathrm{Num} A \subset \mathbb{R}$.

If in addition A is densely defined, then it is hermitian iff $A \subset A^*$.

Remark 3.6.2 *In a part of literature the term "symmetric" is used instead of "hermitian".*

Theorem 3.6.3 *Let* A *be densely defined and hermitian. Then* A *is closable. Besides, one of the following possibilities is true:*

(1) $\mathrm{sp} A \subset \mathbb{R}$.

(2) $\mathrm{sp} A = \{\mathrm{Im} z \geq 0\}$.

(3) $\mathrm{sp}A = \{\mathrm{Im}z \leq 0\}$.

(4) $\mathrm{sp}A = \mathbb{C}$.

Proof. A is closable because $A \subset A^*$ and A^* is closed. $\Box$

Theorem 3.6.4 *Let A be a densely defined operator. Then the following conditions are equivalent:*

(1) $-\mathrm{i}A$ *is the generator of a strongly continuous semigroup of isometries.*

(2) A *is hermitian and* $\mathrm{sp}A \subset \{\mathrm{Im}z \leq 0\}$.

Proof. (1)⇒(2): For $v \in \mathrm{Dom}\, A$,

$$0 = \partial_t(\mathrm{e}^{-\mathrm{i}tA}v|\mathrm{e}^{-\mathrm{i}tA}v)\Big|_{t=0} = -\mathrm{i}(Av|v) + \mathrm{i}(v|Av).$$

Hence A is hermitian.

Isometries are contractions. Hence, by Thm 2.8.1, $\mathrm{sp}A \subset \{\mathrm{Im}z \leq 0\}$.

(2)⇒(1): By Thm 3.4.3, $\|(z + \mathrm{i}A)^{-1}\| \leq |\mathrm{Re}z|^{-1}$, $\mathrm{Re}z > 0$. Hence, by Thm 2.8.1, $\mathrm{e}^{-\mathrm{i}tA}$ is the generator of a strongly continuous contractive semigroup.

For $v \in \mathrm{Dom}\, A$,

$$0 = \partial_t(\mathrm{e}^{-\mathrm{i}tA}v|\mathrm{e}^{-\mathrm{i}tA}v)$$

Hence, for $v \in \mathrm{Dom}\, A$, $\|\mathrm{e}^{-\mathrm{i}tA}v\|^2 = \|v\|^2$. In other words, $\mathrm{e}^{-\mathrm{i}tA}$ is a group of isometries. □

Theorem 3.6.5 *Let A be hermitian. Then the following conditions are equivalent:*

(1) $\mathrm{sp}A \subset \{\mathrm{Im}z \leq 0\}$.

(2) *There exists z_0 with $\mathrm{Im}z_0 > 0$ and $\mathrm{Ran}\,(z_0 - A) = \mathcal{V}$.*

(3) *A is closed and there exists z_0 with $\mathrm{Im}z_0 > 0$ and $\mathrm{Ran}\,(z_0 - A)$ dense in $\mathcal{V}$.*

(4) *A is closed and there exists z_0 with $\mathrm{Im}z_0 > 0$ and $\mathrm{Ker}(\overline{z}_0 - A^*) = \{0\}$.*

3.7 Self-adjoint operators

Definition 3.7.1 *Let A be a densely defined operator on $\mathcal{V}$. A is self-adjoint iff $A^* = A$.*

In other words, A is self-adjoint if for $w \in \mathcal{W}$ there exists $y \in \mathcal{V}$ such that

$$(y|v) = (w|Av),\ v \in \mathrm{Dom}\, A,$$

then $w \in \mathrm{Dom}\, A$ and $Aw = y$.

Theorem 3.7.2 *Every self-adjoint operator is hermitian and closed. If $A \in B(\mathcal{V})$, then it is self-adjoint iff it is hermitian.*

Theorem 3.7.3 *Fix $z_\pm$ with $\pm \mathrm{Im} z_\pm > 0$. Let A be hermitian. Then the following conditions are necessary and sufficient for A to be self-adjoint:*

(1) $\mathrm{sp} A \subset \mathbb{R}$.

(2) $z_\pm \notin \mathrm{sp} A$.

(3) $\mathrm{Ran}\,(z_\pm - A) = \mathcal{V}$.

(4) A *is closed and* $\mathrm{Ran}\,(z_\pm - A)$ *is dense in* $\mathcal{V}$.

(5) A *is closed and* $\mathrm{Ker}(\overline{z}_\pm - A^*) = \{0\}$.

Theorem 3.7.4 *Let $z_0 \in \mathbb{R}$. Let A be hermitian and $z_0 \notin \mathrm{Num} A$. Then the following conditions are sufficient for A to be self-adjoint:*

(1) $z_0 \notin \mathrm{sp} A$.

(2) $\mathrm{Ran}\,(z_0 - A) = \mathcal{V}$.

(3) A *is closed and* $\mathrm{Ran}\,(z_0 - A)$ *is dense in* $\mathcal{V}$.

(4) A *is closed and* $\mathrm{Ker}(z_0 - A^*) = \{0\}$.

Theorem 3.7.5 (Stone Theorem) *Let A be an operator. Then the following conditions are equivalent:*

(1) $\mathrm{i}A$ *is the generator of a strongly continuous group of unitary operators.*

(2) A *is self-adjoint.*

Proof. To prove (1)⇒(2), suppose that $\mathbb{R} \mapsto U(t)$ is a strongly continuous unitary group. Let $-\mathrm{i}A$ be its generator. Then $[0, \infty[\ni t \mapsto U(t), U(-t)$ are semigroups of contractions with the generators $\mathrm{i}A$ and $-\mathrm{i}A$. By Theorem 3.7.5, A is hermitian and $\mathrm{sp}A \subset \mathbb{R}$. Hence A is self-adjoint.

(2)⇒(1): By Theorem 3.7.5 $\pm\mathrm{i}A$ generate semigroups of isometries $\mathrm{e}^{\pm\mathrm{i}tA}$. Clearly, $\mathrm{e}^{\pm\mathrm{i}tA}$ is the inverse of $\mathrm{e}^{\mp\mathrm{i}tA}$. Hence these isometries are unitary. □

3.8 Spectral theorem

Definition 3.8.1 *Recall that $B \in B(\mathcal{V})$ is called normal if $B^*B = BB^*$.*

Let us recall one of the versions of the spectral theorem for bounded normal operators.

Let X be a Borel subset of $\mathbb{C}$. Let $\mathcal{M}(X)$ denote the space of measurable functions on X with values in $\mathbb{C}$. For $f \in \mathcal{M}(X)$ we set $f^*(x) := \overline{f(x)}$, $x \in X$. In particular, the function $X \ni z \mapsto \mathrm{id}(z) := z$ belongs to $\mathcal{M}(X)$.

$\mathcal{L}^\infty(X)$ will denote the space of bounded measurable functions on X.

Theorem 3.8.2 *Let B be a bounded normal operator on $\mathcal{V}$. Then there exists a unique linear map*

$$\mathcal{L}^\infty(\mathrm{sp}B) \ni f \mapsto f(B) \in B(\mathcal{V})$$

such that $1(B) = \mathbb{1}$, $\mathrm{id}(B) = B$, $fg(B) = f(B)g(B)$,
$f(B)^* = f^*(B)$, $\|f(B)\| \leq \sup|f|$,
if $f_n \to f$ pointwise and $|f_n| \leq c$ then $\mathrm{s-}\lim\limits_{n\to\infty} f_n(B) \to f(B)$.
Above, all functions $f, f_n, g \in \mathcal{L}^\infty(\mathrm{sp}B)$.

Theorem 3.8.3 *Let B be a bounded normal operator B. Let $f \in \mathcal{M}(\mathrm{sp}B)$. Set*

$$f_n(x) := \begin{cases} f(x) & |f(x)| \leq n, \\ 0, & |f(x)| > n. \end{cases}$$

$$\mathrm{Dom}(f(B)) \;\; = \;\; \{v \in \mathcal{V} \; : \; \sup \|f_n(B)v\| < \infty\}.$$

Then for $v \in \mathrm{Dom}\, B$ there exists the limit

$$f(B)v \;\; := \;\; \lim_{n\to\infty} f_n(B)v,$$

which defines a closed normal operator.

Let now A be a (possibly unbounded) self-adjoint operator on $\mathcal{V}$.

Theorem 3.8.4 *Then $U := (A + \mathrm{i})(A - \mathrm{i})^{-1}$ is a unitary operator with*

$$\mathrm{sp}U = (\mathrm{sp}^{\mathrm{ext}} A + \mathrm{i})(\mathrm{sp}^{\mathrm{ext}} A - \mathrm{i})^{-1}.$$

Proof. Using the fact that A is hermitian, for $v \in \mathrm{Dom}\, A$ we check that

$$\|(A \pm \mathrm{i})v\|^2 = \|Av\|^2 + \|v\|^2.$$

Therefore, $(A \pm \mathrm{i}) : \mathrm{Dom}\, A \to \mathcal{V}$ are isometric. Using $\mathrm{Ran}\,(A \pm \mathrm{i}) = \mathcal{V}$ we see that they are unitary. Hence so is $(A + \mathrm{i})(A - \mathrm{i})^{-1}$.

The location of the spectrum of U follows from

$$(z - U)^{-1} = (A - \mathrm{i})^{-1}(z - 1)^{-1}\Big(A - \mathrm{i}(z + 1)(z - 1)^{-1}\Big)^{-1}.$$

$\square$

U is unitary, hence normal. If f is a measurable function on $\mathrm{sp}A$, we define

$$f(A) := g(U),$$

where $g(z) = f(\mathrm{i}(z + \mathrm{i})(z - 1)^{-1})$.

Theorem 3.8.5 *The map*

$$\mathcal{M}(\mathrm{sp}A) \ni f \mapsto f(A) \in B(\mathcal{V})$$

is linear and satisfies $1(A) = \mathbb{1}$, $\mathrm{id}(A) = A$, $fg(A) = f(A)g(A)$,
$f(A)^* = f(A)$, $\|f(A)\| \leq \sup|f|$,
where $f, g \in \mathcal{M}(\mathrm{sp}A)$,

Definition 3.8.6 *A possibly unbounded densely defined operator* A *is called normal if* $\mathrm{Dom}\, A = \mathrm{Dom}\, A^*$ *and*

$$\|Av\|^2 = \|A^*v\|, \quad v \in \mathrm{Dom}\, A.$$

One can extend Thm 3.8.5 to normal unbounded operators in an obvious way.

Proposition 3.8.7 *Let* A *be normal. Then the closure of the numerical range is the convex hull of its spectrum.*

Proof. We can write $A = \int \lambda \mathrm{d}E(\lambda)$, where $E(\lambda)$ is a spectral measure. Then for $\|v\| = 1$, $(v|Av)$ is the center of mass of the measure $(v|\mathrm{d}E(\lambda)v)$. $\square$

3.9 Essentially self-adjoint operators

Definition 3.9.1 *An operator* $A : \mathcal{V} \to \mathcal{V}$ *is essentially self-adjoint iff* A^{cl} *is self-adjoint.*

Theorem 3.9.2 (1) *Every essentially self-adjoint operator is hermitian and closable.*

(2) A *is essentially self-adjoint iff* A^* *is self-adjoint.*

Theorem 3.9.3 *Let* A *be hermitian. Fix* $z_\pm \in \mathbb{C}$ *with* $\pm \mathrm{Im} z_\pm > 0$. *Then the following conditions are necessary and sufficient for* A *to be essentially self-adjoint:*

(1) $\mathrm{Ran}\,(z_+ - A)$ *and* $\mathrm{Ran}\,(z_- - A)$ *are dense in* $\mathcal{V}$.

(2) $\mathrm{Ker}(\overline{z}_+ - A^*) = \{0\}$ *and* $\mathrm{Ker}(\overline{z}_- - A^*) = \{0\}$.

Theorem 3.9.4 *Let* A *be hermitian. Let* $z_0 \in \mathbb{R} \backslash \mathrm{Num} A$. *Then the following conditions are sufficient for* A *to be essentially self-adjoint:*

(1) $\mathrm{Ran}\,(z_0 - A)$ *is dense in* $\mathcal{V}$.

(2) $\mathrm{Ker}(z_0 - A^*) = \{0\}$.

3.10 Rigged Hilbert space

Let $\mathcal{V}$ be a Hilbert space with the scalar product $(\cdot|\cdot)$. Suppose that T is a self-adjoint operator on $\mathcal{V}$ with $T \geq c_0 > 0$. Then $\mathrm{Dom}\, T$ can equipped with the scalar product

$$(Tv|Tw), \quad v, w \in \mathrm{Dom}\, T$$

is a Hilbert space embedded in $\mathcal{V}$. We will prove a converse construction, that leads from an embedded Hilbert space to a positive self-adjoint operator.

Let $\mathcal{V}^*$ denote the space of bounded antilinear functionals on $\mathcal{V}$. The Riesz lemma says that $\mathcal{V}^*$ is a Hilbert space naturally isomorphic to $\mathcal{V}$.

Suppose that $\mathcal{W}$ is a Hilbert space contained and dense in $\mathcal{V}$. We assume that for $c_0 > 0$

$$(w|w)_{\mathcal{W}} \geq c_0(w|w), \ w \in \mathcal{W}. \tag{3.10.2}$$

Of course, $\mathcal{W}^*$ is also a Hilbert naturally isomorphic to $\mathcal{W}$. However, we do not want to use this isomorphism.

Let $J : \mathcal{W} \to \mathcal{V}$ denote the embedding. By (3.10.2), it is bounded. Clearly $J^* : \mathcal{V} \to$

$\mathcal{W}^*$ (where we use the identification $\mathcal{V} \simeq \mathcal{V}^*$). We have $\mathrm{Ker} J^* = (\mathrm{Ran}\, J)^\perp = \{0\}$ and $(\mathrm{Ran}\, J^*)^\perp = \mathrm{Ker} J = \{0\}$. Hence J^* is a dense embedding of $\mathcal{V}$ in $\mathcal{W}^*$. Thus we obtain a triplet of Hilbert spaces, sometimes called a rigged Hilbert space

$$\mathcal{W} \subset \mathcal{V} \subset \mathcal{W}^*.$$

Theorem 3.10.1 *There exists a unique positive injective self-adjoint operator T on $\mathcal{V}$ such that $\mathrm{Dom}\, T = \mathcal{W}$ and*

$$(w_1|w_2)_{\mathcal{W}} = (Tw_1|Tw_2), \quad w_1, w_2 \in \mathcal{W}. \tag{3.10.3}$$

Proof. Without loss of generality we will assume that $c_0 = 1$.

For $v \in \mathcal{V}$, $w \in \mathcal{W}$, we have

$$|(w|v)| \leq \|w\|\|v\| \leq \|w\|_{\mathcal{W}}\|v\|.$$

By the Riesz lemma, there exists $A : \mathcal{V} \to \mathcal{W}$ such that

$$(w|v) = (w|Av)_{\mathcal{W}}, \tag{3.10.4}$$

We treat A as an operator from $\mathcal{V}$ to $\mathcal{V}$. A is bounded, because

$$\|Av\|^2 \leq \|Av\|_{\mathcal{W}}^2 = (Av|Av)_{\mathcal{W}} = (Av|v) \leq \|Av\|\|v\|.$$

A is positive, (and hence in particular self-adjoint) because

$$(Av|v) = (Av|Av)_{\mathcal{W}} \geq 0.$$

A has a zero kernel, because $Av = 0$ implies

$$0 = (w|Av)_{\mathcal{V}} = (w|v), \quad v \in \operatorname{Dom} \mathcal{W},$$

and $\mathcal{W}$ is dense.

Thus $T := A^{-1/2}$ defines a positive self-adjoint operator $\geq \mathbb{1}$. We have

$$(w|y)_{\mathcal{W}} = (w|T^2 y), \quad w \in \mathcal{W}, \quad y \in \operatorname{Dom} T^2 = \operatorname{Ran} A.$$

Using the lemma below, with two embedded Hilbert spaces $\mathcal{W}$ and $\operatorname{Dom} T$ having a common dense subspace $\operatorname{Dom} T^2$, we obtain $\mathcal{W} = \operatorname{Dom} T$ and the equality (3.10.3). $\square$

Lemma 3.10.2 *Let $\mathcal{W}_+, \mathcal{W}_-$ be two Hilbert spaces embedded in a Hilbert space $\mathcal{V}$. Suppose that their norms satisfy*

$$\|w\| \leq \|w\|_+, \ \ w \in \mathcal{W}_+, \quad \|w\| \leq \|w\|_-, \ \ w \in \mathcal{W}_-.$$

Let $\mathcal{D} \subset \mathcal{W}_+ \cap \mathcal{W}_-$ be dense both in $\mathcal{W}_+$ and in $\mathcal{W}_-$. Suppose $\|\cdot\|_+ = \|\cdot\|_-$ in $\mathcal{D}$. Then $\mathcal{W}_+ = \mathcal{W}_-$ and $\|\cdot\|_+ = \|\cdot\|_-$.

Proof. Let $w_+ \in \mathcal{W}_+$. There exists $(w_n) \subset \mathcal{D}$ such that $\|w_n - w_+\|_+ \to 0$. This implies $\|w_n - w_+\| \to 0$.

Besides w_n is Cauchy in $\mathcal{W}_-$ Hence there exists $w_- \in \mathcal{W}_-$ such that $\|w_n - w_-\|_- \to 0$.

This implies $\|w_n - w_-\| \to 0$. Hence $w_+ = w_-$. Besides, $\|w_+\|_+ = \lim \|w_n\|_+ = \lim \|w_n\|_- = \|w_-\|_-$.

Thus $\mathcal{W}_+ \subset \mathcal{W}_-$ and in $\mathcal{W}_+$ the norm $\|\cdot\|_+$ coincides with the norm $\|\cdot\|_-$. $\square$

By functional calculus for self-adjoint operators we can define $S := T^2$. Clearly, $T = \sqrt{S}$ and

$$(v|Sw) = (v|w)_{\mathcal{W}}, \quad v \in \mathrm{Dom}\,\sqrt{S}, \quad w \in \mathrm{Dom}\,S.$$

We will say that the operator S is associated with the sesquilinear form $(\cdot|\cdot)_{\mathcal{W}}$.

3.11 Polar decomposition

Let A be a densely defined closed operator. Let $S+1$ be the positive operator associated with the sesquilinear form

$$(Av|Aw) + (v|w), \quad v, w \in \mathrm{Dom}\,A.$$

Theorem 3.11.1 *$S = A^*A$.*

In order to prove this theorem, introduce $\mathcal{V}_1 = (\mathbb{1}+T)^{-1}\mathcal{V}$ and $\mathcal{V}_{-1} = (\mathbb{1}+T)\mathcal{V}$, so that $\mathcal{V}_1 = \mathrm{Dom}\, A$ and $\mathcal{V}_{-1} = \mathcal{V}_1^*$. Denote by $A_{(1)}$ the operator A treated as an operator $\mathcal{V}_1 \to \mathcal{V}$. Clearly, $A_{(1)}$ is bounded, and so is $A_{(1)}^* : \mathcal{V} \to \mathcal{V}_{-1}$.

Proposition 3.11.2 (1) $\mathrm{Dom}\, A^* = \{v \in \mathcal{V} \ : \ A_{(1)}^* v \in \mathcal{V}\}$.

(2) *On* $\mathrm{Dom}\, A^*$ *the operators* A^* *and* $A_{(1)}^*$ *coincide.*

(3) $\mathrm{Dom}\, T^2 = \{v \in \mathrm{Dom}\, A \ : \ Av \in \mathrm{Dom}\, A^*\}$

(4) *For* $v \in \mathrm{Dom}\, T^2$, $T^2 v = A^* A v$.

Proof. (1). Let $w \in \mathcal{V}$. We have

$$w \in \mathrm{Dom}\, A^* \ \Leftrightarrow \ |(w|Av)| \leq C\|v\|, \quad v \in \mathrm{Dom}\, A. \tag{3.11.5}$$

But $\mathrm{Dom}\, A = \mathcal{V}_1$ and $(w|Av) = (A_{(1)}^* w|v)$. Hence, (3.11.5) is equivalent to

$$|(A_{(1)}^* w|v)| \leq C\|v\|, \quad v \in \mathrm{Dom}\, A, \tag{3.11.6}$$

which means $A_{(1)}^* w \in \mathcal{V}$.

In the proof of (3) we will use the operators $T_{(1)}$ and $T_{(1)}^*$ defined analogously as $A_{(1)}$ and $A_{(1)}^*$. We have

$$T_{(1)}^* T_{(1)} = A_{(1)}^* A_{(1)}. \tag{3.11.7}$$

In fact, for $v, w \in \mathcal{V}_1$

$$(w|T_{(1)}^* T_{(1)} v) = (T_{(1)} w | T_{(1)} v) = (A_{(1)} w | A_{(1)} v) = (w | A_{(1)}^* A_{(1)} v).$$

Now

$$\begin{aligned}
\mathrm{Dom}\, T^2 &= \{v \in \mathcal{V}_1 \ : \ T_{(1)}^* T_{(1)} v \in \mathcal{V}\} \ \text{ by spectral theorem} \\
&= \{v \in \mathcal{V}_1 \ : \ A_{(1)}^* A_{(1)} v \in \mathcal{V}\} \ \text{ by (3.11.7)} \\
&= \{v \in \mathcal{V}_1 \ : \ A_{(1)} v \in \mathrm{Dom}\, A^*\} \ \text{ by (1).}
\end{aligned}$$

□

Theorem 3.11.3 *Let A be closed. Then there exist a unique positive operator $|A|$ and a unique partial isometry U such that* $\mathrm{Ker} U = \mathrm{Ker} A$ *and* $A = U|A|$. *We have then* $\mathrm{Ran}\, U =$

$\mathrm{Ran}\, A^{\mathrm{cl}}$.

Proof. The operator A^*A is positive. By the spectral theorem, we can then define

$$|A| := \sqrt{A^*A}.$$

On $\mathrm{Ran}\,|A|$ the operator U is defined by

$$U\ |A|v := Av.$$

It is isometric, because

$$\||A|v\|^2 = (v||A|^2v) = (v|A^*Av) = \|Av\|^2,$$

and correctly defined. We can extend it to $(\mathrm{Ran}\,|A|)^{\mathrm{cl}}$ by continuity. On $\mathrm{Ker}|A| = (\mathrm{Ran}\,|A|)^{\mathrm{cl}}$, we extend it by putting $Uv = 0$. $\square$

3.12 Scale of Hilbert spaces I

Let A be a positive self-adjoint operator on $\mathcal{V}$ with $A \geq 1$. We define the family of Hilbert spaces $\mathcal{V}_\alpha$, $\alpha \in \mathbb{R}$ as follows.

For $\alpha \geq 0$, we set $\mathcal{V}_\alpha := \mathrm{Ran}\, A^{-\alpha} = \mathrm{Dom}\, A^\alpha$ with the scalar product

$$(v|w)_\alpha := (v|A^{2\alpha}w).$$

Clearly, for $0 \leq \alpha \leq \beta$ we have the embedding $\mathcal{V}_\alpha \supset \mathcal{V}_\beta$.

For $\alpha \leq 0$ we set $\mathcal{V}_\alpha := \mathcal{V}_{-\alpha}^*$, If $\alpha \leq \beta \leq 0$ we have a natural inclusion $\mathcal{V}_\alpha \supset \mathcal{V}_\beta$.

Note that we have the identification $\mathcal{V} = \mathcal{V}^*$, hence both definitions give $\mathcal{V}_0 = \mathcal{V}$.

Thus we obtain

$$\mathcal{V}_\alpha \supset \mathcal{V}_\beta, \text{ for any } \alpha \leq \beta. \tag{3.12.8}$$

Note that for $\alpha \leq 0$ $\mathcal{V}$ is embedded in $\mathcal{V}_\alpha$ and for $v, w \in \mathcal{V}$

$$(v|w)_\alpha = \left(v|A^{2\alpha}w\right).$$

Moreover, $\mathcal{V}$ is dense in $\mathcal{V}_\alpha$.

Sometimes we will use a different notation: $A^{-\alpha}\mathcal{V} = \mathcal{V}_\alpha$.

By restriction or extension, we can reinterpret the operator A^β as a unitary operator

$$A^\beta_{(-\alpha)} : A^\alpha\mathcal{V} \to A^{\alpha+\beta}\mathcal{V}.$$

If B is a self-adjoint operator, then we will use the notation $\langle B\rangle := (1 + B^2)^{1/2}$. Clearly, B gives rise to a bounded operator

$$B_{(\alpha)} : \langle B\rangle^{-\alpha}\mathcal{V} \to \langle B\rangle^{-\alpha+1}\mathcal{V}.$$

Thus every self-adjoint operator can be interpreted in many ways, depending on β we choose. The standard choice corresponding to $\beta = 1$

$$B_{(1)} : \mathrm{Dom}\, B = \langle B\rangle^{-1}\mathcal{V} \to \mathcal{V}$$

can be called the "operator interpretation".

Another interpretation is often useful:

$$B_{(1/2)} : \langle B\rangle^{-1/2}\mathcal{V} \to \langle B\rangle^{1/2}\mathcal{V},$$

the "form interpretation". One often introduces the form domain $\mathcal{Q}(B) := \langle B\rangle^{-1/2}\mathcal{V}$. We obtain a sesquilinear form

$$\mathcal{Q}(B) \times \mathcal{Q}(B) \ni (v, w) \mapsto (v|B_{(1/2)}w).$$

3.13 Scale of Hilbert spaces II

We will write $A > 0$ if A is positive, self-adjoint and $\mathrm{Ker}A = \{0\}$. One can generalize the definition of the scale of spaces $A^\alpha\mathcal{V}$ to the case $A > 0$.

Set $\mathcal{V}_+ := \mathrm{Ran}\,\mathbb{1}_{[1,\infty[}(A)$, $\mathcal{V}_- := \mathrm{Ran}\,\mathbb{1}_{[0,1[}(A)$. Let $A_\pm := A\Big|_{\mathcal{V}_\pm}$. Then $A_+ \geq 1$ and $A_-^{-1} \geq 1$. Hence we can define the scales of spaces $A_+^\alpha\mathcal{V}_+$, $A_-^\alpha\mathcal{V}_- := (A_-^{-1})^{-\alpha}\mathcal{V}_-$, $\alpha \in \mathbb{R}$. We set

$$A^\alpha\mathcal{V} := A_+^\alpha\mathcal{V}_+ \oplus A_-^\alpha\mathcal{V}_-. \tag{3.13.9}$$

If A is not bounded away from zero, then the scale (3.13.9) does not have the nested property (3.12.8). However, for any $\alpha, \beta \in \mathbb{R}$, $A^\alpha \mathcal{V} \cap A^\beta \mathcal{V}$ is dense in $A^\alpha \mathcal{V}$. Again, we have a family of unitary operators

$$A_{(\alpha)}^\beta : A^\alpha \mathcal{V} \to A^{\alpha+\beta} \mathcal{V}.$$

3.14 Complex interpolation

Let us recall a classic fact from complex analysis:

Theorem 3.14.1 (Three lines theorem) *Suppose that a function* $\{0 \leq \mathrm{Re} z \leq 1\} \ni z \mapsto f(z) \in \mathbb{C}$ *is continuous, bounded, analytic in the interor of its domain, and satisfies the bounds*

$$\begin{aligned} |f(\mathrm{i}s)| &\leq c_0, \\ |f(1+\mathrm{i}s)| &\leq c_1, \quad s \in \mathbb{R}. \end{aligned} \tag{3.14.10}$$

Then

$$|f(t+\mathrm{i}s)| \leq c_0^{1-t} c_1^t, \quad t \in [0,1],\ s \in \mathbb{R}. \tag{3.14.11}$$

Theorem 3.14.2 *Let $A > 0$ on $\mathcal{V}$, $B > 0$ on $\mathcal{W}$. Consider an operator $C : \mathcal{V} \cap A^{-1}\mathcal{V} \to \mathcal{W} \cap B^{-1}\mathcal{W}$ that satisfies*

$$\begin{aligned} \|Cv\| &\leq c_0\|v\|, \\ \|BCv\| &\leq c_1\|Av\|, \quad v \in \mathcal{V} \cap A^{-1}\mathcal{V}. \end{aligned}$$

(In other words, C is bounded as an operator $\mathcal{V} \to \mathcal{W}$ with the norm $\leq c_0$ and $A^{-1}\mathcal{V} \to B^{-1}\mathcal{W}$ with the norm $\leq c_1$.) Then, for $0 \leq t \leq 1$,

$$\|B^t Cv\| \leq c_0^{1-t} c_1^t \|A^t v\|, \tag{3.14.12}$$

and so C extends to a bounded operator

$$C : A^{-t}\mathcal{V} \to B^{-t}\mathcal{W},$$

with the norm $\leq c_0^{1-t} c_1^t$.

Proof. Let $w \in \mathcal{W} \cap B^{-1}\mathcal{W}$ and $v \in \mathcal{V} \cap A^{-1}\mathcal{V}$. The vector valued functions $z \mapsto B^z w$ and $z \mapsto A^z v$ are bounded on $\{0 \le \mathrm{Re} z \le 1\}$, and hence so is

$$f(z) := (B^{\overline{z}} w | C A^z v)$$

We have

$$\begin{aligned} |f(\mathrm{i}s)| &\le c_0 \|w\| \|v\|, \\ |f(1+\mathrm{i}s)| &\le c_1 \|w\| \|v\|, \quad s \in \mathbb{R}. \end{aligned}$$

Hence,

$$|f(t)| \le c_0^{1-t} c_1^t \|w\| \|v\|, \quad t \in [0,1].$$

This implies (3.14.12), by the density of $\mathcal{W} \cap B^{-1}\mathcal{W}$. $\square$

3.15 Relative operator boundedness

Let A be a closed operator and B an operator with $\mathrm{Dom}\, B \supset \mathrm{Dom}\, A$. Recall that the (operator) A-bound of B is

$$a_1 := \inf_{\nu>0} \sup_{v\neq 0,\ v\in\mathrm{Dom}\, A} \left(\frac{\|Bv\|^2}{\|Av\|^2+\nu^2\|v\|^2}\right)^{\frac{1}{2}}. \tag{3.15.13}$$

In a Hilbert space

$$\|Av\|^2+\nu^2\|v\|^2 = \|(A^*A+\nu^2)^{1/2}v\|^2.$$

Therefore, (3.15.13) can be rewritten as

$$a_1 = \inf_{\nu>0} \|B(A^*A+\nu^2)^{-1/2}\|. \tag{3.15.14}$$

If, moreover, A is self-adjoint, then, using the unitarity of $(A^2+\nu^2)^{-1/2}(\pm \mathrm{i}\nu - A)$, we can rewrite (3.15.14) as

$$a_1 = \inf_{\nu\neq 0} \|B(\mathrm{i}\nu - A)^{-1}\|. \tag{3.15.15}$$

Using Prop. 1.7.4 we obtain

$$a_1 = \inf_{z\in \mathrm{rs} A} \|B(z-A)^{-1}\|. \tag{3.15.16}$$

Theorem 3.15.1 (Kato-Rellich) *Let A be self-adjoint, B hermitian. Let B be A-bounded with the $A-$bound <1. Then*

(1) $A+B$ *is self-adjoint on* $\mathrm{Dom}\, A$*.*

(2) *If A is essentally self-adjoint on $\mathcal{D}$, then $A+B$ is essentially self-adjoint on $\mathcal{D}$.*

Proof. Clearly, $A+B$ is hermitian on $\mathrm{Dom}\, A$. Moreover, for some ν, $\|B(\pm \mathrm{i}\nu - A)^{-1}\| < 1$. Hence, $\mathrm{i}\nu - A - B$ and $-\mathrm{i}\nu - A - B$ are invertible. □

3.16 Relative form boundedness

Assume first that A is a positive self-adjoint operator. Let B be a bounded operator from $\mathrm{Dom}\, A^{1/2} = (1\!\!1 + A)^{-1/2}\mathcal{V}$ to $(1\!\!1 + A)^{1/2}\mathcal{V}$. Note that B defines a bounded quadratic form on $\mathcal{Q}(B) := (1\!\!1 + A)^{-1/2}\mathcal{V}$

$$\mathcal{Q}(B) \ni u, v \mapsto (u|Bv).$$

Let us assume that this form is hermitian, that is

$$(u|Bv) = \overline{(v|Bu)}.$$

Definition 3.16.1 *We say that B is form-bounded relatively to A iff there exist constants a, b such that*

$$|(v|Bv)| \leq a(v|Av) + b(v|v), \quad v \in \mathrm{Dom}\, A^{1/2}. \tag{3.16.17}$$

The infimum of a satisfying (3.16.17) is called the A-bound of B.

In other words: the A-form bound of B equals

$$a_2 := \inf_{c>0} \sup_{v \in \mathrm{Dom}\, A^{1/2} \setminus \{0\}} \frac{(v|Bv)}{(v|Av) + c(v|v)}.$$

This can be rewritten as

$$a_2 = \inf_{c>0} \|(A+c)^{-1/2} B (A+c)^{-1/2}\|.$$

Theorem 3.16.2 *A is a positive self-adjoint operator. Let B have the form A-bound less* *than* 1*. Then*

$$R(\mu) := \sum_{j=0}^{\infty} (\mu - A)^{-1/2} \big((\mu - A)^{-1/2} B (\mu - A)^{-1/2}\big)^j (\mu - A)^{-1/2}$$

is convergent for large negative μ*. Moreover,* $R(z)$ *is a resolvent of a self-adjoint bounded from below operator, which will be called the form sum of* A *and* B *and denoted, by the abuse of notation,* $A + B$*. We have* $\mathrm{Dom}\,|A + B|^{\frac{1}{2}} = \mathrm{Dom}\,|A|^{\frac{1}{2}}$*.*

We can generalize the concept of the form boundedness to the context of not necessarily positive operators as follows. Let A be a self-adjoint operator. Let B be a bounded operator from $\langle A\rangle^{-1/2}\mathcal{V}$ to $\langle A\rangle^{1/2}\mathcal{V}$. We assume that the form given by B is hermitian.

Definition 3.16.3 *The improved form* A*-bound of* B *is*

$$a_2' := \inf_{\nu>0,\mu} \|(A - \mu)^2 + \nu^2)^{-\frac{1}{4}} B ((A - \mu)^2 + \nu^2)^{-\frac{1}{4}}\|. \qquad (3.16.18)$$

(3.16.18) can be rewritten as

$$a_2' = \inf_{\nu>0,\mu} \|(\mu + \mathrm{i}\nu - A)^{-\frac{1}{2}} B(\mu + \mathrm{i}\nu - A)^{-\frac{1}{2}}\|. \tag{3.16.19}$$

Theorem 3.16.4 *Let A be a self-adjoint operator. Let B have the improved A-form bound less than 1. Then there exists open subsets in the upper and lower complex half-plane such that the series*

$$R(z) := \sum_{j=0}^{\infty} (z-A)^{-1/2} \big((z-A)^{-1/2} B (z-A)^{-1/2}\big)^j (z-A)^{-1/2}$$

is convergent. Moreover, $R(z)$ is a resolvent of a self-adjoint operator, which will be called the form sum of A and B and denoted, by the abuse of notation, $A + B$.

The form boundedness is stronger than the operator boundedness. Indeed, suppose that B is a hermitian operator on $\mathcal{V}$ with $\mathrm{Dom}\, B \supset \mathrm{Dom}\, A$ and

$$\|B\big((A-\mu)^2 + \nu^2\big)^{1/2}\| \leq a.$$

This means that B is bounded as an operator $\big((A-\mu)^2+\nu^2\big)^{-1/2}\mathcal{V}\to\mathcal{V}$ and as an operator $\mathcal{V}\to\big((A-\mu)^2+\nu^2\big)^{1/2}\mathcal{V}$, in both cases with norm $\leq a$. By the complex interpolation, it is bounded as an operator $\big((A-\mu)^2+\nu^2\big)^{-1/4}\mathcal{V}\to\big((A-\mu)^2+\nu^2\big)^{1/4}\mathcal{V}$ with norm $\leq a$. In particular, we have $a_2'\leq a_1$, where a_1 is the operator A-bound and a_2' is the improved form A-bound.

3.17 Self-adjointness of Schrödinger operators

The following lemma is a consequence of the Hölder inequality:

Lemma 3.17.1 *Let* $1\leq p,q\leq\infty$ *and* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. *Then the operator of multiplication by* $V\in L^p(\mathbb{R}^d)$ *is bounded as a map* $L^q(\mathbb{R}^d)\to L^r(\mathbb{R}^d)$ *with norm equal to* $\|V\|_q$.

The following two lemmas follow from the Hardy-Littlewood-Sobolev inequality:

Lemma 3.17.2 *The operator* $(\mathbb{1}-\Delta)^{-1}$ *is bounded from* $L^2(\mathbb{R}^d)$ *to* $L^q(\mathbb{R}^d)$ *in the following cases:*

(1) *For* $d=1,2,3$ *if* $\frac{1}{\infty}\leq\frac{1}{q}\leq\frac{1}{2}$.

(2) *For* $d = 4$ *if* $\frac{1}{\infty} < \frac{1}{q} \leq \frac{1}{2}$.

(3) *For* $d \geq 5$ *if* $\frac{1}{2} - \frac{2}{d} \leq \frac{1}{q} \leq \frac{1}{2}$.

Lemma 3.17.3 *The operator* $(\mathbb{1} - \Delta)^{-\frac{1}{2}}$ *is bounded from* $L^2(\mathbb{R}^d)$ *to* $L^q(\mathbb{R}^d)$ *in the following cases:*

(1) *For* $d = 1$ *if* $\frac{1}{\infty} \leq \frac{1}{q} \leq \frac{1}{2}$.

(2) *For* $d = 2$ *if* $\frac{1}{\infty} < \frac{1}{q} \leq \frac{1}{2}$.

(3) *For* $d \geq 3$ *if* $\frac{1}{2} - \frac{1}{d} \leq \frac{1}{q} \leq \frac{1}{2}$.

Proposition 3.17.4 *Let* $V \in L^p + L^\infty(\mathbb{R}^d)$*,where*

(1) *for* $d = 1, 2, 3$*,* $p = 2$*,*

(2) *for* $d = 4$*,* $p > 2$*,*

(3) *for* $d \geq 5$*,* $p = \frac{d}{2}$.

Then the $-\Delta$*-bound of* V *is zero. Hence* $-\Delta + V(x)$ *is self-adjoint on* $\mathrm{Dom}(-\Delta)$.

Proof. We need to show that

$$\lim_{c\to\infty} V(x)(c-\Delta)^{-1} = 0, \tag{3.17.20}$$

where (3.17.20) is understood as an operator on $L^2(\mathbb{R}^d)$.

For any $\epsilon > 0$ we can write $V = V_\infty + V_p$, where $V_\infty \in L^\infty(\mathbb{R}^d)$, $V_p \in L^p(\mathbb{R}^d)$ and $\|V_p\|_p \leq \epsilon$. Now

$$V(x)(c-\Delta)^{-1} \;=\; V_\infty(x)(c-\Delta)^{-1} + V_p(x)(c-\Delta)^{-1}.$$

The first term has the norm $\leq \|V_\infty\|_\infty c^{-1}$. Consider the second term. Let

$$\frac{1}{q} + \frac{1}{p} = \frac{1}{2}$$

$\|V_p(x)_{L^q\to L^2} = \|V_p\|_p \leq \epsilon$, and $\|(c-\Delta)^{-1}_{L^2\to L^q}\|$ is uniformly finite for $c > 1$ by Lemma 3.17.3.
□

Proposition 3.17.5 *Let $V \in L^p + L^\infty(\mathbb{R}^d)$,where*

(1) *for* $d = 1$, $p = 1$,

(2) *for* $d = 2$, $p > 1$,

(3) *for* $d \geq 3$, $p = \frac{d}{2}$.

Then the form $-\Delta$*-bound of* V *is zero. Hence* $-\Delta + V(x)$ *can be defned in the sense of the form sum with the form domain* $\mathrm{Dom}(\sqrt{-\Delta})$.

Proof. We need to show that

$$\lim_{c\to\infty} (c - \Delta)^{-1/2} V(x)(c - \Delta)^{-1/2} = 0, \tag{3.17.21}$$

where (3.17.21) is understood as an operator on $L^2(\mathbb{R}^d)$. For any $\epsilon > 0$ we can write $V = V_\infty + V_p$, where $V_\infty \in L^\infty(\mathbb{R}^d)$, $V_p \in L^p(\mathbb{R}^d)$ and $\|V_p\|_p \leq \epsilon$. Now

$$\begin{aligned}(c - \Delta)^{-1/2} V(x)(c - \Delta)^{-1/2} \;=\; & (c - \Delta)^{-1/2} V_\infty(x)(c - \Delta)^{-1/2} \\ & + \big(|V_p(x)|^{1/2}(c - \Delta)^{-1/2}\big)^* \mathrm{sgn} V_p(x) |V_p(x)|^{1/2}(c - \Delta)^{-1/2}.\end{aligned}$$

The first term has the norm $\leq \|V_\infty\|_\infty c^{-1}$. Consider the second term. Let

$$\frac{1}{q} + \frac{2}{p} = \frac{1}{2}.$$

$\| |V_p(x)|^{1/2}_{L^q(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \| = \sqrt{\|V_p\|_p} \leq \sqrt{\epsilon}$ and $\|(c - \Delta)^{-1/2}_{L^2 \to L^q}\|$ is uniformly finite for $c > 1$ by Lemma 3.17.3. $\square$

Chapter 4

Positive forms

4.1 Quadratic forms

Let $\mathcal{V}, \mathcal{W}$ be complex vector spaces.

Definition 4.1.1 $\mathfrak{a}$ *is called a sesquilinear form on* $\mathcal{W} \times \mathcal{V}$ *iff it is a map*

$$\mathcal{W} \times \mathcal{V} \ni (w, v) \mapsto \mathfrak{a}(w, v) \in \mathbb{C}$$

antilinear wrt the first argument and linear wrt the second argument.

If $\lambda \in \mathbb{C}$, then λ can be treated as a sesquilinear form $\lambda(w, v) := \lambda(w|v)$. If $\mathfrak{a}$ is a form, then we define $\lambda\mathfrak{a}$ by $(\lambda\mathfrak{a})(w, v) := \lambda\mathfrak{a}(w, v)$. and $\mathfrak{a}^*$ by $\mathfrak{a}^*(v, w) := \overline{\mathfrak{a}(w, v)}$. If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are forms, then we define $\mathfrak{a}_1 + \mathfrak{a}_2$ by $(\mathfrak{a}_1 + \mathfrak{a}_2)(w, v) := \mathfrak{a}_1(w, v) + \mathfrak{a}_2(w, v)$.

Suppose that $\mathcal{V} = \mathcal{W}$. We will write $\mathfrak{a}(v) := \mathfrak{a}(v, v)$. We will call it a quadratic form. The knowledge of $\mathfrak{a}(v)$ determines $\mathfrak{a}(w, v)$:

$$\mathfrak{a}(w, v) = \frac{1}{4}\left(\mathfrak{a}(w + v) + \mathrm{i}\mathfrak{a}(w - \mathrm{i}v) - \mathfrak{a}(w - v) - \mathrm{i}\mathfrak{a}(w + \mathrm{i}v)\right). \qquad (4.1.1)$$

Suppose now that $\mathcal{V}, \mathcal{W}$ are Hilbert spaces. A form is bounded iff

$$|\mathfrak{a}(w, v)| \leq C\|w\|\|v\|.$$

Proposition 4.1.2 (1) *Let $\mathfrak{a}$ be a bounded sesquilinear form on $\mathcal{W} \times \mathcal{V}$. Then there exists a unique operator $A \in B(\mathcal{V}, \mathcal{W})$ such that*

$$\mathfrak{a}(w, v) = (w|Av).$$

(2) *If* $A \in B(\mathcal{V}, \mathcal{W})$*, then* $(w|Av)$ *is a bounded sesquilinear form on* $\mathcal{W} \times \mathcal{V}$*.*

Proof. (2) is obvious. To show (1) note that $w \mapsto \mathfrak{a}(w|v)$ is an antilinear functional on $\mathcal{W}$. Hence there exists $\eta \in \mathcal{W}$ such that $\mathfrak{a}(w, v) = (w|\eta)$. We put $Av := \eta$.

Theorem 4.1.3 *Suppose that* $\mathcal{D}, \mathcal{Q}$ *are dense linear subspaces of* $\mathcal{V}, \mathcal{W}$ *and* $\mathfrak{a}$ *is a bounded sesquilinear form on* $\mathcal{D} \times \mathcal{Q}$*. Then there exists a unique extension of* $\mathfrak{a}$ *to a bounded form on* $\mathcal{V} \times \mathcal{W}$*.*

4.2 Sesquilinear quasiforms

Let $\mathcal{V}, \mathcal{W}$ be complex spaces. We say that $\mathfrak{t}$ is a sesquilinear quasiform on $\mathcal{W} \times \mathcal{V}$ iff there exist subspaces $\mathrm{Dom}_\mathrm{l}\, \mathfrak{t} \subset \mathcal{W}$ and $\mathrm{Dom}_\mathrm{r}\, \mathfrak{t} \subset \mathcal{V}$ such that

$$\mathrm{Dom}_\mathrm{l}\mathfrak{t} \times \mathrm{Dom}_\mathrm{r}\mathfrak{t} \ni (w, v) \mapsto \mathfrak{t}(w, v) \in \mathbb{C}$$

is a sesquilinear map. From now on by a sesquilinear form we will mean a sesquilinear quasiform.

We define a form $\mathfrak{t}^*$ with the domains $\mathrm{Dom}_\mathrm{l}\, \mathfrak{t}^* := \mathrm{Dom}_\mathrm{r}\, \mathfrak{t}$, $\mathrm{Dom}_\mathrm{r}\, \mathfrak{t}^* := \mathrm{Dom}_\mathrm{l}\, \mathfrak{t}$, by the formula

$\mathfrak{t}^*(v,w) := \overline{\mathfrak{t}(w,v)}$. If $\mathfrak{t}_1$ are $\mathfrak{t}_2$ forms, then we define $\mathfrak{t}_1+\mathfrak{t}_2$ with the domain $\mathrm{Dom}_\mathrm{l}(\mathfrak{t}_1+\mathfrak{t}_2) := \mathrm{Dom}_\mathrm{l}\,\mathfrak{t}_1 \cap \mathrm{Dom}_\mathrm{l}\,\mathfrak{t}_1$, $\mathrm{Dom}_\mathrm{r}(\mathfrak{t}_1+\mathfrak{t}_2) := \mathrm{Dom}_\mathrm{r}\,\mathfrak{t}_1 \cap \mathrm{Dom}_\mathrm{r}\,\mathfrak{t}_1$ by $(\mathfrak{t}_1+\mathfrak{t}_2)(w,v) := \mathfrak{t}_1(w,v)+\mathfrak{t}_2(w,v)$. We write $\mathfrak{t}_1 \subset \mathfrak{t}_2$ if $\mathrm{Dom}_\mathrm{l}\,\mathfrak{t}_1 \subset \mathrm{Dom}_\mathrm{l}\,\mathfrak{t}_2$, $\mathrm{Dom}_\mathrm{r}\,\mathfrak{t}_1 \subset \mathrm{Dom}_\mathrm{r}\,\mathfrak{t}_2$, and $\mathfrak{t}_1(w,v) = \mathfrak{t}_2(w,v)$, $w \in \mathrm{Dom}_\mathrm{l}\,\mathfrak{t}_1$, $v \in \mathrm{Dom}_\mathrm{r}\,\mathfrak{t}_1$.

From now on, we will usually assume that $\mathcal{W} = \mathcal{V}$ and $\mathrm{Dom}_\mathrm{l}\,\mathfrak{t} = \mathrm{Dom}_\mathrm{r}\,\mathfrak{t}$ and the latter subspace will be simply denoted by $\mathrm{Dom}\,\mathfrak{t}$. We will then write $\mathfrak{t}(v) := \mathfrak{t}(v,v)$, $v \in \mathrm{Dom}\,\mathfrak{t}$.

The numerical range of the form $\mathfrak{t}$ is defined as

$$\mathrm{Num}\mathfrak{t} := \{\mathfrak{t}(v)\ :\ v \in \mathrm{Dom}\,\mathfrak{t},\ \|v\| = 1\}.$$

We proved that $\mathrm{Num}\mathfrak{t}$ is a convex set.

With every operator T on $\mathcal{V}$ we can associate the form

$$\mathfrak{t}_1(w,v) := (w|Tv), \quad w,v \in \mathrm{Dom}\,T.$$

Clearly, $\mathrm{Num}\mathfrak{t}_1 = \mathrm{Num}T$. If T is self-adjoint, we will however prefer to associate a different form to it, see Theorem 4.5.1.

The form $\mathfrak{t}$ is bounded iff $\mathrm{Num}\mathfrak{t}$ is bounded. Equivalently, $|\mathfrak{t}(v)| \leq c\|v\|^2$.

$\mathfrak{t}$ is hermitian iff $\mathrm{Num}\mathfrak{t} \subset \mathbb{R}$. An equivalent condition: $\mathfrak{t}(w,v) = \overline{\mathfrak{t}(v,w)}$.

A form $\mathfrak{t}$ is bounded from below, if there exists c such that

$$\mathrm{Num}\mathfrak{t} \subset \{z \ : \ \mathrm{Re} z > c\}.$$

A form $\mathfrak{t}$ is positive if $\mathrm{Num}\mathfrak{t} \subset [0,\infty[$. In this section we develop the basics of the theory of positive forms.

Note that many of the concepts and facts about positive forms generalize to hermitian bounded from below forms. In fact, if $\mathfrak{t}$ is bounded from below hermitian, then for some $c \in \mathbb{R}$ we have a positive form $\mathfrak{t} + c$. We leave these generalizations to the reader.

4.3 Closed positive forms

Let $\mathfrak{s}$ be a positive form.

Definition 4.3.1 *We say that $\mathfrak{s}$ is a closed form iff $\mathrm{Dom}\,\mathfrak{s}$ with the scalar product*

$$(w|v)_{\mathfrak{s}} := (\mathfrak{s}+1)(w,v), \quad w,v \in \mathrm{Dom}\,\mathfrak{s}, \tag{4.3.2}$$

is a Hilbert space. We will then write $\|v\|_{\mathfrak{s}} := \sqrt{(v|v)_{\mathfrak{s}}}$.

Clearly, the scalar product (4.3.2) is equivalent with

$$(\mathfrak{s}+c)(w,v), \quad w,v \in \mathrm{Dom}\,\mathfrak{s},$$

for any $c > 0$.

Theorem 4.3.2 *The form $\mathfrak{s}$ is closed iff for any sequence (v_n) in $\mathrm{Dom}\,\mathfrak{s}$, if $v_n \to v$ and $\mathfrak{s}(v_n - v_m) \to 0$, then $v \in \mathrm{Dom}\,\mathfrak{s}$ and $\mathfrak{s}(v_n - v) \to 0$.*

Example 4.3.3 *Let A be an operator. Then*

$$(Aw|Av), \quad w,v \in \mathrm{Dom}\,A,$$

is a closed form iff A is closed.

4.4 Closable positive forms

Let $\mathfrak{s}$ be a positive form.

Definition 4.4.1 *We say that $\mathfrak{s}$ is a closable form iff there exists a closed form $\mathfrak{s}_1$ such that $\mathfrak{s} \subset \mathfrak{s}_1$.*

Theorem 4.4.2 (1) *The form $\mathfrak{s}$ is closable $\Leftrightarrow$ for any sequence $(v_n) \subset \mathrm{Dom}\,\mathfrak{s}$, if $v_n \to 0$ and $\mathfrak{s}(v_n - v_m) \to 0$, then $\mathfrak{s}(v_n) \to 0$.*

(2) *If $\mathfrak{s}$ is closable, then there exists the smallest closed form $\mathfrak{s}_1$ such that $\mathfrak{s} \subset \mathfrak{s}_1$. We will denote it by $\mathfrak{s}^{\mathrm{cl}}$.*

(3) $\mathrm{Num}\mathfrak{s}$ *is dense in* $\mathrm{Num}\mathfrak{s}^{\mathrm{cl}}$

Proof. (1) $\Rightarrow$ follows immediately from Theorem 4.3.2.

To prove (1) $\Leftarrow$, define $\mathfrak{s}_1$ as follows: $v \in \mathrm{Dom}\,\mathfrak{s}_1$, iff there exists a sequence $(v_n) \subset \mathrm{Dom}\,\mathfrak{s}$ such that $v_n \to v$ and $\mathfrak{s}(v_n - v_m) \to 0$. From $\mathfrak{s}(v_n) \leq (\sqrt{\mathfrak{s}(v_1)} + \sqrt{\mathfrak{s}(v_n - v_1)})^2$ it follows that $\big(\mathfrak{s}(v_n)\big)$ is bounded. From $|\mathfrak{s}(v_n) - \mathfrak{s}(v_m)| \leq \sqrt{\mathfrak{s}(v_n - v_m)}\big(\sqrt{\mathfrak{s}(v_n)} + \sqrt{\mathfrak{s}(v_n)}\big)$ it follows that $\big(\mathfrak{s}(v_n)\big)$ is a Cauchy sequence. Hence we can set $\mathfrak{s}_1(v) := \lim_{n\to\infty} \mathfrak{s}(v_n)$

To show that the definition is correct, suppose that $(w_n) \in \mathrm{Dom}\,\mathfrak{s}$, $w_n \to v$ and $\mathfrak{s}(w_n - w_m) \to 0$. Then $\mathfrak{s}(v_n - w_n - (v_m - w_m)) \to 0$ and $v_n - w_n \to 0$. By the hypothesis we get $\mathfrak{s}(v_n - w_n) \to 0$. Hence, $\lim\limits_{n\to\infty} \mathfrak{s}(v_n) = \lim\limits_{n\to\infty} \mathfrak{s}(w_n)$. Thus the definition of $\mathfrak{s}_1$ does not depend on the choice of the sequence v_n. It is clear that $\mathfrak{s}_1$ is a closed form containing $\mathfrak{s}$. Hence $\mathfrak{s}$ is closable.

To prove (2) note that the form $\mathfrak{s}_1$ constructed above is the smallest closed form containg $\mathfrak{s}$. □

Example 4.4.3 *Let A be an operator. Then*

$$(Aw|Av), \quad w, v \in \mathrm{Dom}\, A,$$

is closable iff A is a closable operator. Then

$$(A^{\mathrm{cl}}w|A^{\mathrm{cl}}v), \quad w, v \in \mathrm{Dom}\, A^{\mathrm{cl}}$$

is its closure.

Definition 4.4.4 *We say that a linear subspace $\mathcal{Q}$ is an essential domain of the form $\mathfrak{s}$ if*

$$\left(\mathfrak{s}\Big|_{\mathcal{Q}\times\mathcal{Q}}\right)^{\mathrm{cl}} = \mathfrak{s}.$$

4.5 Operators associated with positive forms

Let S be a self-adjoint operator. We define the form $\mathfrak{s}$ as follows:

$$\mathfrak{s}(v,w) := (|S|^{1/2}v|\mathrm{sgn}(S)|S|^{1/2}w), \quad v,w \in \mathrm{Dom}\,\mathfrak{s} := \mathrm{Dom}\,|S|^{1/2}.$$

We will say that $\mathfrak{s}$ is the form associated with the operator S.

Theorem 4.5.1 (1) $\mathrm{Num}S$ *is dense in* $\mathrm{Num}\mathfrak{s}$*.*

(2) *If S is positive, then $\mathfrak{s}$ is a closed positive form and* $\mathrm{Dom}\,S$ *is its essential domain.*

The next theorem describes the converse construction. It follows immediately from Thm 3.11.2.

Theorem 4.5.2 (Lax-Milgram Theorem) *Let $\mathfrak{s}$ be a densely defined closed positive form.*

Then there exists a unique positive self-adjoint operator S such that

$$\mathfrak{s}(v,w) := (S^{1/2}v|S^{1/2}w), \quad v,w \in \mathrm{Dom}\,\mathfrak{s} := \mathrm{Dom}\,S^{1/2}.$$

Proof. By Thm 3.10.1 applied to $\mathrm{Dom}\,\mathfrak{s}$ there exists a positive self-adjoint operator T such that

$$\mathfrak{s}(v,w) := (Tv|Tw), \quad v,w \in \mathrm{Dom}\,\mathfrak{s} := \mathrm{Dom}\,T.$$

We set $S := T^2$. □

We will say that S is the operator associated with the form $\mathfrak{s}$.

4.6 Perturbations of positive forms

Theorem 4.6.1 *Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be positive forms.*

(1) $\mathfrak{t}_1 + \mathfrak{t}_2$ *is also a positive form.*

(2) *If $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are closed, then $\mathfrak{t}_1 + \mathfrak{t}_2$ is closed as well.*

(3) *If* $\mathfrak{t}_1$ *and* $\mathfrak{t}_2$ *are closable, then* $\mathfrak{t}_1 + \mathfrak{t}_2$ *is closable as well and* $(\mathfrak{t}_1 + \mathfrak{t}_2)^{\rm cl} \subset \mathfrak{t}_1^{\rm cl} + \mathfrak{t}_2^{\rm cl}$.

Definition 4.6.2 *Let* $\mathfrak{p}$, $\mathfrak{t}$ *be hermitian forms. Let* $\mathfrak{t}$ *be positive. We say that* $\mathfrak{p}$ *is* $\mathfrak{t}$*-bounded iff* $\mathrm{Dom}\,\mathfrak{t} \subset \mathrm{Dom}\,\mathfrak{p}$ *and*

$$b := \inf_{c>0} \sup_{v\in\mathrm{Dom}\,\mathfrak{t}} \frac{|\mathfrak{p}(v)|}{\mathfrak{t}(v) + c\|v\|^2} < \infty.$$

The number b *is called the* $\mathfrak{t}$*-bound of* $\mathfrak{p}$.

Theorem 4.6.3 *Let* $\mathfrak{t}$ *be positive and let* $\mathfrak{p}$ *be* $\mathfrak{t}$*-bounded with the* $\mathfrak{t}$*-bound* < 1. *Then*

(1) *The form* $\mathfrak{t} + \mathfrak{p}$ *(with the domain* $\mathrm{Dom}\,\mathfrak{t}$*) is bounded from below.*

(2) $\mathfrak{t}$ *is closed* $\Leftrightarrow$ $\mathfrak{t} + \mathfrak{p}$ *is closed.*

(3) $\mathfrak{t}$ *is closable* $\Leftrightarrow$ $\mathfrak{t} + \mathfrak{p}$ *is closable, and then* $\mathrm{Dom}(\mathfrak{t} + \mathfrak{p})^{\rm cl} = \mathrm{Dom}\,\mathfrak{t}^{\rm cl}$.

Proof. Let us prove (1). For some $b < 1$, we have

$$(\mathfrak{t} + \mathfrak{p})(v) \geq \mathfrak{t}(v) - |\mathfrak{p}(v)| \geq (1 - b)\mathfrak{t}(v) - c\|v\|^2. \tag{4.6.3}$$

This proves that $\mathfrak{t} + \mathfrak{p}$ is bounded from below.

To see (2) and (3), note that (4.6.3) and

$$(1+b)\mathfrak{t}(v)+c\|v\|^2 \geq (\mathfrak{t}+\mathfrak{p})(v)$$

prove that the norms $\|\cdot\|_{\mathfrak{t}}$ and $\|\cdot\|_{\mathfrak{t}+\mathfrak{p}}$ are equivalent. $\square$

4.7 Friedrichs extensions

Theorem 4.7.1 *Let T be a positive densely defined operator. Then the form*

$$\mathfrak{t}(w,v) := (w|Tv), \quad w,v \in \operatorname{Dom}\mathfrak{t} := \operatorname{Dom} T$$

is closable.

Proof. Suppose that $w_n \in \operatorname{Dom} T$, $w_n \to 0$, $\lim\limits_{n,m\to\infty} \mathfrak{t}(w_n - w_m) = 0$. Then

$$\begin{aligned} |\mathfrak{t}(w_n)| &\leq |\mathfrak{t}(w_n - w_m, w_n)| + |\mathfrak{t}(w_m, w_n)| \\ &\leq \sqrt{\mathfrak{t}(w_n)}\sqrt{\mathfrak{t}(w_n - w_m)} + (w_m|Tw_n). \end{aligned}$$

For any $\epsilon > 0$ there exists N such that for $n, m > N$ we have $\mathfrak{t}(w_n - w_m) \leq \epsilon^2$. Besides, $\lim_{m\to\infty}(w_m|Tw_n) = 0$. Therefore, for $n > N$,

$$|\mathfrak{t}(w_n)| \leq \epsilon|\mathfrak{t}(w_n)|^{1/2}.$$

Hence $\mathfrak{t}(w_n) \to 0$. $\square$

Thus there exists a unique postive self-adjoint operator T^{Fr} associated with the form $\mathfrak{t}^{\mathrm{cl}}$. The operator T^{Fr} is called the Friedrichs extension of T.

Clearly, $\mathrm{Dom}\, T$ is then essential form domain of T^{Fr}. However in general it is not an essential operator domain of T^{Fr}. The theorem says nothing about essential operator domains.

For example, consider any open $\Omega \subset \mathbb{R}^d$. Note that $C_{\mathrm{c}}^\infty(\Omega)$ is dense in $L^2(\Omega)$. The equation

$$(f| - \Delta g) = \int \overline{\nabla f(x)} \nabla g(x) \mathrm{d}x, \quad f \in C_{\mathrm{c}}^\infty(\Omega)$$

shows that $-\Delta$ on $C_{\mathrm{c}}^\infty(\Omega)$ is a positive operator. Its Friedrichs extension is called the laplacian on Ω with the Dirichlet boundary conditions.

If V is any positive bounded from below function we can consider $\Delta + V(x)$ and define its Friedrichs extension.

Chapter 5

Non-maximal operators

5.1 Defect indices

If $\mathcal{V}$ is a finite dimensional Hilbert space and $\mathcal{V}_1, \mathcal{V}_2$ its two subspaces such that $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$, then we have the following obvious inequalities:

$$\begin{aligned} \dim \mathcal{V}_1 + \dim \mathcal{V}_2 &\leq \dim \mathcal{V}, \\ \dim \mathcal{V}_1 &\leq \dim \mathcal{V}_2^{\perp}, \\ \dim \mathcal{V}_2 &\leq \dim \mathcal{V}_1^{\perp}. \end{aligned}$$

If $\dim \mathcal{V} = \infty$, then clearly the first inequality loses its interest. However the other two inequalities, which are still true, may be interesting.

Let A be an operator on a Hilbert space $\mathcal{V}$.

Theorem 5.1.1 $\dim \mathrm{Ran}\,(z - A)^{\perp} = \dim \mathrm{Ker}(\overline{z} - A^*)$ *is a constant function on connected components of* $\mathbb{C}\backslash(\mathrm{Num}A)^{\mathrm{cl}}$.

Proof. Let us show that if $|z - z_1| < \mathrm{dist}(z, \mathrm{Num}A)$, then

$$\mathrm{Ran}\,(z - A) \cap \mathrm{Ran}\,(z_1 - A)^{\perp} = \{0\}. \tag{5.1.1}$$

Let $w \in \mathrm{Ran}\,(z - A)$. Then there exists $v \in \mathrm{Dom}\,A$ such that

$$w = (z - A)v$$

and $\|v\| \leq c\|w\|$, where $c = \big(\mathrm{dist}(z, \mathrm{Num}A)\big)^{-1}$. If moreover, $w \in \mathrm{Ran}\,(z_1 - A)^{\perp} =$

$\mathrm{Ker}(\overline{z}_1 - A^*)$, then

$$\begin{aligned} 0 &= \big((z_1 - A^*)w|v\big) \\ &= \big(w|(z-A)v\big) + (z_1 - z)(w|v) \\ &= \|w\|^2 + (z - z_1)(w|v). \end{aligned}$$

But

$$\big|\|w\|^2 + (z_1 - z)(w|v)\big| \geq (1 - |z_1 - z|c)\|w\|^2 > 0,$$

which is a contradiction and completes the proof of (5.1.1).

Now (5.1.1) implies that $\dim \mathrm{Ran}\,(z-A)^\perp \leq \dim \mathrm{Ran}\,(z_1 - A)^\perp$. □

5.2 Extensions of hermitian operators

Let A be closed hermitian.

Theorem 5.2.1 *The so-called defect indices of A*

$$n_\pm := \dim \mathrm{Ker}(z - A^*),\ \ z \in \mathbb{C}_\pm$$

do not depend on z. Then A possesses a self-adjoint extension iff $n_+ = n_-$. Moreover, one of the following possibilities is true:

(1) $\mathrm{Num}A \neq \mathbb{R}$.

   (i) $\mathrm{sp}A \subset \mathbb{R}$, $n_+ = n_- = 0$ *and A is self-adjoint.*

   (ii) $\mathrm{sp}A = \mathbb{C}$, $n_+ = n_- > 0$.

(2) $\mathrm{Num}A = \mathbb{R}$.

   (i) $\mathrm{sp}A \subset \mathbb{R}$, $n_+ = n_- = 0$, *A is self-adjoint.*

   (ii) $\mathrm{sp}A = \{\mathrm{Im}z \geq 0\}$, $n_+ > 0$, $n_- = 0$, *A is not self-adjoint.*

   (iii) $\mathrm{sp}A = \{\mathrm{Im}z \leq 0\}$, $n_+ = 0$, $n_- > 0$, *A is not self-adjoint.*

   (iv) $\mathrm{sp}A = \mathbb{C}$, $n_+ > 0$, $n_- > 0$, *A is not self-adjoint.*

Proof. The existence of self-adjoint extensions for $n_+ = n_-$ follows from Theorem 5.2.3.

The remaining statements are essentially a special case of Theorem 5.1.1. □

Definition 5.2.2 *Define on* $\mathrm{Dom}\, A^*$ *the following scalar product:*

$$(v|w)_{A^*} := (v|w) + (A^*v|A^*w)$$

and the following antihermitian form:

$$[v|w]_{A^*} := (A^*v|w) - (v|A^*w).$$

The A^*-*closedness and the* A^*-*orthogonality is defined using the scalar product* $(\cdot|\cdot)_{A^*}$.

Theorem 5.2.3 (1) *Every closed extension of* A *is a restriction of* A^* *to an* A^*-*closed subspace in* $\mathrm{Dom}\, A^*$ *containing* $\mathrm{Dom}\, A$.

(2)

$$\mathrm{Dom}\, A^* = \mathrm{Dom}\, A \oplus \mathrm{Ker}(A^* + \mathrm{i}) \oplus \mathrm{Ker}(A^* - \mathrm{i})$$

and the components in the above direct sum are A^**-closed,* A^*-*orthogonal and*

$$\begin{aligned}
(w_0 \oplus w_+ \oplus w_-|v_0 \oplus v_+ \oplus v_-)_{A^*} &= (w_0|v_0) + (Aw_0|Av_0) + 2(w_+|v_+) + 2(w_-|v_-), \\
[w_0 \oplus w_+ \oplus w_-|v_0 \oplus v_+ \oplus v_-]_{A^*} &= 2\mathrm{i}(w_+|v_+) - 2\mathrm{i}(w_-|v_-).
\end{aligned}$$

Proof. (1) is obvious. In (2) the A^*-orthogonality and the A^*-closedness are easy.

Let $w \in \mathrm{Dom}\, A^*$ and

$$w \perp_{A^*} \mathrm{Dom}\, A \oplus \mathrm{Ker}(A^* + \mathrm{i}).$$

In particular, for $v \in \mathrm{Dom}\, A$ we have

$$0 = (A^*w|A^*v) + (w|v) = (A^*w|Av) + (w|v).$$

Hence $A^*w \in \mathrm{Dom}\, A^*$ and

$$A^*A^*w = -w.$$

Therefore,

$$(A^* + \mathrm{i})(A^* - \mathrm{i})w = 0.$$

Thus

$$(A^* - \mathrm{i})w \in \mathrm{Ker}(A^* + \mathrm{i}). \tag{5.2.2}$$

If $y \in \mathrm{Ker}(A^* + \mathrm{i})$, then

$$\mathrm{i}(y|(A^* - \mathrm{i})w) = (A^*y|A^*w) + (y|w) = (y|w)_{A^*} = 0$$

In particular, by (5.2.2) we can set $y = (A^* - \mathrm{i})w$. We get $w \in \mathrm{Ker}(A^* - \mathrm{i})$. $\Box$

$\mathrm{Dom}\, A$ belongs to the kernel of the antisymmetric form $[\cdot,\cdot]_{A^*}$. Therefore, in what follows we restrict this form to

$$\mathcal{V}_{\mathrm{def}} := \mathrm{Ker}(A^* + \mathrm{i}) \oplus \mathrm{Ker}(A^* - \mathrm{i}).$$

We will write

$$\mathcal{Z}^{\mathrm{per}} := \{v \in \mathcal{V}_{\mathrm{def}} \ : \ [z, v]_{A^*} = 0, \ z \in \mathcal{Z}\}.$$

We will say that a subspace $\mathcal{Z}$ of $\mathcal{V}_{\mathrm{def}}$ is A^*-isotropic iff $[\cdot|\cdot]_{A^*}$ vanishes on $\mathcal{Z}$ and A^*-Lagrangian if $\mathcal{Z}^{\mathrm{per}} = \mathcal{Z}$.

Every A^*-closed subspace of $\mathcal{V}$ containing $\mathrm{Dom}\, A$ is of the form $\mathrm{Dom}\, A \oplus \mathcal{Z}$, where $\mathcal{Z} \subset \mathcal{V}_{\mathrm{def}}$. If

$$A \subset B \subset A^*,$$

then the subspace $\mathcal{Z}$ corresponding to B will be denoted by $\mathcal{Z}_B$.

Theorem 5.2.4 (1) *We have*

$$\mathcal{Z}_{B^*} = (\mathcal{Z}_B)^{\mathrm{per}}.$$

(2) B *is hermitian iff* $\mathcal{Z}_B$ *is* A^*-*isotropic iff there exists a partial isometry* $U : \mathrm{Ker}(A^* + \mathrm{i}) \to \mathrm{Ker}(A^* - \mathrm{i})$ *such that*

$$\mathcal{Z} := \{w_+ \oplus U w_+ \ : \ w_+ \in \mathrm{Ran}\, U^* U\}.$$

(3) B *is self-adjoint iff* $\mathcal{Z}_B$ *is* A^**-Lagrangian iff there exists a unitary* $U : \mathrm{Ker}(A^* + \mathrm{i}) \to \mathrm{Ker}(A^* - \mathrm{i})$ *such that*

$$\mathcal{Z} := \{w_+ \oplus U w_+ \ : \ w_+ \in \mathrm{Ker}(A^* + \mathrm{i})\}.$$

5.3 Extension of positive operators

(This subsection is based on unpublished lectures of S.L.Woronowicz).

Theorem 5.3.1 *Let $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$ and*

$$B = \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix}$$

be an operator in $B(\mathcal{V})$ with B_{11} invertible. Then B is positive iff $B_{11} \geq 0$, $B_{01} = B_{10}^*$ and $B_{00} \geq B_{01}B_{11}^{-1}B_{10}$.

Proof. Let $v_0 \in \mathcal{V}_0$, $v_1 \in \mathcal{V}_1$. For $v_z = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}$. Then

$$\begin{aligned} 0 \leq (v|Bv) &= (v_0 B_{00} v_0) + (v_0|B_{01}v_1) + (v_1|B_{10}v_0) + (v_1|B_{11}v_1) \\ &= \big(v_0|(B_{00} - B_{01}B_{11}^{-1}B_{10})v_0\big) + \|B_{11}^{-1/2}B_{10}v_0 + B_{11}^{1/2}v_1\|^2 \end{aligned}$$

This proves $\Rightarrow$.

Let us prove $\Leftarrow$. The necessity of $B_{11} \geq 0$ is obvious. Given v_0, we can choose $v_1 = -B_{11}^{-1}B_{10}v_0$. This shows that $B_{00} - B_{01}B_{11}^{-1}B_{10}$ has to be positive. $\Box$

Suppose that G is hermitian, positive and closed. We would like to describe its positive self-adjoint extensions. Thus we are looking for positive self-adjoint H such that $G \subset H$.

The operator $G + \mathbb{1}$ is injective and has a closed range. Define $\mathcal{V}_1 := \operatorname{Ran} G$ and set $\mathcal{V}_0 := \mathcal{V}_1^\perp$, so that $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$. Let $A \in B(\mathcal{V}_1, \mathcal{V})$ be the left inverse of $G + \mathbb{1}$. We can write it as

$$A = \begin{bmatrix} A_{01} \\ A_{11} \end{bmatrix}$$

We are looking for a bounded operator

$$(\mathbb{1} + H)^{-1} = B = \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} \in B(\mathcal{V})$$

that extends A and $0 \leq B \leq \mathbb{1}$. Clearly, $B_{11} = A_{11}$, $B_{01} = A_{01}$, $B_{10} = A_{01}^*$. By Theorem 5.3.1,

$$\begin{aligned} B_{00} &\geq B_{01} B_{11}^{-1} B_{10}, \\ \mathbb{1}_{00} - B_{00} &\geq B_{01} (\mathbb{1}_{11} - B_{11})^{-1} B_{10}. \end{aligned}$$

Thus we can choose any $B_{00} \in B(\mathcal{V}_0)$ satisfying

$$\mathbb{1}_{00} - A_{01}(\mathbb{1}_{11} - A_{11})^{-1}A_{01}^* \geq B_{00} \geq A_{01}A_{11}^{-1}A_{01}^*.$$

This condition has two extreme solutions: The smallest $A_{01}A_{11}^{-1}A_{01}^*$ yields the largest extension, called the Friedrichs extension H^{Fr}. The largest $\mathbb{1}_{00} - A_{01}(\mathbb{1}_{11} - A_{11})^{-1}A_{01}^*$, gives the smallest positive extension, called the Krein extension H^{Kr}. We have the following formula for both extensions:

$$\begin{aligned}
&(\mathbb{1} + H^{\mathrm{Fr}})^{-1} \\
:= \;& \big(A_{11}^{1/2} + A_{01}A_{11}^{-1/2}\big)\big(A_{11}^{1/2} + A_{01}A_{11}^{-1/2}\big)^*, \\
&\mathbb{1} - (\mathbb{1} + H^{\mathrm{Kr}})^{-1} \\
:= \;& \big((\mathbb{1}_{11} - A_{11})^{1/2} - A_{01}(\mathbb{1}_{11} - A_{11})^{-1/2}\big)\big((\mathbb{1}_{11} - A_{11})^{1/2} - A_{01}(\mathbb{1}_{11} - A_{11})^{-1/2}\big)^*.
\end{aligned}$$

Chapter 6

Aronszajn-Donoghue Hamiltonians and their renormalization

6.1 Construction

Recall that the operators $(h|$ and $|h)$ are defined by

$$\begin{aligned} \mathcal{H} \ni v &\mapsto (h|v := (h|v) \in \mathbb{C}, \\ \mathbb{C} \ni \alpha &\mapsto |h)\alpha := \alpha h \in \mathcal{H}. \end{aligned} \tag{6.1.1}$$

In particular, $|h)(h|$ equals the orthogonal projection onto h times $\|h\|^2$.

Let H_0 be a self-adjoint operator on $\mathcal{H}$, $h \in \mathcal{H}$ and $\lambda \in \mathbb{R}$.

$$H_\lambda := H_0 + \lambda|h)(h|, \tag{6.1.2}$$

is a rank one perturbation of H_0. We will call (6.1.2) the Aronszajn Donoghue Hamiltonian.

We would like to describe how to define the Aronszajn-Donoghue Hamiltonian if h is not necessarily a bounded functional on $\mathcal{H}$. It will turn out that it is natural to consider 3 types of h:

$$\text{I. } h \in \mathcal{H}, \qquad \text{II. } h \in \langle H_0\rangle^{1/2}\mathcal{H} \setminus \mathcal{H}, \qquad \text{III. } h \in \langle H_0\rangle\mathcal{H} \setminus \langle H_0\rangle^{1/2}\mathcal{H}, \tag{6.1.3}$$

where $\langle H_0\rangle := (1 + H_0^2)^{1/2}$.

Clearly, in the case I H_λ is self-adjoint on $\mathrm{Dom}\, H_0$. We will see that in the case II one can easily define H_λ as a self-adjoint operator, but its domain is no longer equal to $\mathrm{Dom}\, H_0$. In the case III, strictly speaking, the formula (6.1.2) does not make sense. Nevertheless, it is possible to define a renormalized Aronszajn-Donoghue Hamiltonian. To do this one needs to renormalize the parameter λ. This procedure resembles the renormalization of the charge in quantum field theory. In this case usually the parameter λ looses its meaning, so we will abandon the notation H_λ. Instead, one can label the Hamiltonian by various parameters, which we will put in brackets.

Lemma 6.1.1 *In Case I with $\lambda \neq 0$, the resolvent of H equals*

$$\begin{aligned} R(z) &:= (z-H)^{-1} \\ &= (z-H_0)^{-1} - g(z)^{-1}(z-H_0)^{-1}|h)(h|(z-H_0)^{-1}, \end{aligned} \tag{6.1.4}$$

where

$$g(z) := -\lambda^{-1} + (h|(z-H_0)^{-1}h). \tag{6.1.5}$$

defined for $z \notin \mathrm{sp} H_0$.

Proof. We have

$$\begin{aligned} R(z) - (z-H_0)^{-1} &= \lambda R(z)|h)(h|(z-H_0)^{-1} \\ &= \lambda (z-H_0)^{-1}|h)(h|R(z). \end{aligned} \tag{6.1.6}$$

Hence the range of (6.1.6) is $\mathbb{C}(z-H_0)^{-1}h$, and the kernel is $\{(z-H_0)^{-1}h\}^{\perp}$. Therefore, (6.1.6) has the form

$$-g(z)^{-1}(z-H_0)^{-1}|h)(h|(z-H_0)^{-1} \tag{6.1.7}$$

for some complex function $g(z)$. Thus it remains to determine $g(z)$ in (6.1.4). We insert (6.1.4) into

$$\lambda(z-H_0)^{-1}|h)(h|R(z) \;=\; -g(z)^{-1}(z-H_0)^{-1}|h)(h|(z-H_0)^{-1},$$

and we obtain the formula for g, sometimes called Krein's formula. □

For $\lambda = 0$, clearly

$$R_0(z) = (z-H_0)^{-1}. \tag{6.1.8}$$

The following theorem describes how to define the Aronszajn-Donoghue Hamiltonian also in cases II and III:

Theorem 6.1.2 *Assume that:*

(A) $h \in \langle H_0\rangle^{1/2}\mathcal{H}$, $\lambda \in \mathbb{R}\cup\{\infty\}$. *Let* $R_\lambda(z)$ *be given by (6.1.8) or (6.1.4) with* $g_\lambda(z)$ *given*

by (6.1.5),

or

(B) $h \in \langle H_0\rangle\mathcal{H}$, $\gamma \in \mathbb{R}$. *Let* $R_{(\gamma)}(z)$ *be given by (6.1.4) where* $g_{(\gamma)}(z)$ *is the solution of*

$$\begin{cases} \partial_z g_{(\gamma)}(z) = -\left(h|(z-H_0)^{-2}h\right), \\ \frac{1}{2}\left(g_{(\gamma)}(\mathrm{i}) + g_{(\gamma)}(-\mathrm{i})\right) = \gamma. \end{cases} \tag{6.1.9}$$

Then, for $z \in \mathbb{C}\backslash \mathrm{sp}H_0$ *such that* $g(z) \neq 0$

(1) $z \mapsto R(z)$ *is a pseudoresolvent (a function with values in bounded operators that fulfill the first resolvent formula);*

(2) $\mathrm{Ker}R(z) = \{0\}$, *unless* $h \in \mathcal{H}$ *and* $\lambda = \infty$*;*

(3) $\mathrm{Ran}\, R(z)$ *is dense in* $\mathcal{H}$, *unless* $h \in \mathcal{H}$ *and* $\lambda = \infty$*;*

(4) $R(z)^* = R(\overline{z})$.

Hence, except for the case $h \in \mathcal{H}$, $\lambda = \infty$, *there exists a unique densely defined self-adjoint operator* H *such that* $R(z)$ *is the resolvent of* H.

The initial condition in (6.1.9) can be called the renormalization condition. It is easy to solve

(6.1.9) obtaining

$$g_{(\gamma)}(z) = \gamma + \big(h|\big((z - H_0)^{-1} + H_0(1 + H_0^2)^{-1}\big)h\big) \,.$$

If $g(\beta) = 0$ and $\beta \notin \mathrm{sp}H_0$, then H has an eigenvalue at β, and the corresponding eigenprojection is

$$1_{\{\beta\}}(H) = (h|(\beta - H_0)^{-2}h)^{-1}(\beta - H_0)^{-1}|h)(h|(\beta - H_0)^{-1}.$$

In Case I and II the function $\mathbb{R} \cup \{\infty\} \ni \lambda \mapsto H_\lambda$ is increasing.

In Case III we rename H_0 as $H_{(\infty)}$.

6.2 Cut-off method

Another way to define H for the case $h \in \langle H_0 \rangle \mathcal{H}$ is the cut-off method. For $\Lambda > 0$ we define

$$h_\Lambda := 1\!\!1_{[-\Lambda,\Lambda]}(H_0)\, h, \tag{6.2.10}$$

where $1\!\!1_{[-\Lambda,\Lambda]}(H_0)$ is the spectral projection for H_0 onto $[-\Lambda, \Lambda] \subset \mathbb{R}$. Note that $h_\Lambda \in \mathcal{H}$.

We fix the running coupling constant by

$$-\lambda_\Lambda^{-1} := \gamma + (h_\Lambda | H_0(1+H_0^2)^{-1} h_\Lambda)$$

and set the cut-off Hamiltonian to be

$$H_\Lambda := H_0 + \lambda_\Lambda |h_\Lambda)(h_\Lambda|. \tag{6.2.11}$$

Then the resolvent for H_Λ is given by

$$R_\Lambda(z) = (z-H_0)^{-1} - g_\Lambda(z)^{-1}(z-H_0)^{-1}|h_\Lambda)(h_\Lambda|(z-H_0)^{-1}, \tag{6.2.12}$$

where

$$g_\Lambda(z) := -\lambda_\Lambda^{-1} + \left(h_\Lambda|(z-H_0)^{-1}h_\Lambda\right). \tag{6.2.13}$$

Note that λ_Λ is chosen in such a way that the renormalization condition $\frac{1}{2}\left(g_\Lambda(\mathrm{i}) + g_\Lambda(-\mathrm{i})\right) = \gamma$. holds. The cut-off Hamiltonian converges to the renormalized Hamiltonian:

Theorem 6.2.1 *Assume that $h \in \langle H_0\rangle\mathcal{H}$. Then $\lim_{k\to\infty} R_\Lambda(z) = R(z)$.*

6.3 Extensions of hermitian operators

Let H_0 be as above and $h \in \langle H_0 \rangle \mathcal{H} \backslash \mathcal{H}$. (Thus we consider jointly Case II and III.) Define $H_{\min}$ to be the restriction of H_0 to

$$\mathrm{Dom}(H_{\min}) := \{v \in \mathrm{Dom}(H_0) = \langle H_0 \rangle^{-1} \mathcal{H} \ : \ (h|v) = 0\}.$$

Then $H_{\min}$ is a closed densely defined Hermitian operator. Set $H_{\max} := H_{\min}^*$. Then

$$\mathrm{Dom}(H_{\max}) = \mathrm{Span}\big(\,\mathrm{Dom}\, H_0 \cup \{(z_0 - H_0)^{-1} h\}\big),$$

where $z_0 \in \mathrm{rs} H_0$. Note that $\mathrm{Ker}(H_{\max} \pm \mathrm{i})$ is spanned by

$$v_{\pm} := (\pm \mathrm{i} - H_0)^{-1} h.$$

Thus the indices of defect of $H_{\min}$ are $(1, 1)$.

The operators $H_{(\gamma)}$ described in the previous subsection are self-adjoint extensions of $H_{\min}$.

To obtain $H_{(\gamma)}$ it suffices to increase the domain of $H_{\min}$ by adding the vector

$$\frac{\gamma + (h|H_0(1+H_0^2)^{-1}h)}{\gamma - \mathrm{i}(h|(1+H_0^2)^{-1}h)}(\mathrm{i}-H_0)^{-1}h - \frac{\gamma + (h|H_0(1+H_0^2)^{-1}h)}{\gamma + \mathrm{i}(h|(1+H_0^2)^{-1}h)}(\mathrm{i}+H_0)^{-1}h,$$

If $H_{(\gamma)}$ has an eigenvalue β outside of $\mathrm{sp}H_0$, then instead we can add the vector

$$(\beta - H_0)^{-1}h.$$

6.4 Positive H_0

Let us consider the special case $H_0 > 0$.

We can define the positive form $\mathfrak{h}_{\min}$ associated with $H_{\min}$:

$$\begin{aligned} \mathfrak{h}_{\min}(v,v) &= (v|H_{\min}v) = (v|H_0v), \\ v &\in \mathrm{Dom}(\mathfrak{h}_{\min}) := \mathrm{Dom}\, H_{\min} = \{v \in \mathrm{Dom}(H_0) \ : \ (h|v) = 0\}. \end{aligned}$$

In Case II and III the form $\mathfrak{h}_{\min}$ is densely defined.

Clearly, g is analytic on $\mathbb{C}\backslash[0,\infty[$. g restricted to $]-\infty,0[$ is a decreasing function (in all cases I, II and III). Therefore, H can possess at most one negative eigenvalue.

We distinguish subcases of Cases I, II and III

Case I iff $h \in \mathcal{H}$;

$$\begin{aligned} &\text{Case Ia} \quad \text{iff} \quad h \in \mathrm{Dom}\, H_0^{-1/2}; \\ &\text{Case Ib} \quad \text{iff} \quad h \notin \mathrm{Dom}\, H_0^{-1/2}. \end{aligned}$$

Case II iff $h \in (1 + H_0)^{1/2}\mathcal{H}$, $h \notin \mathcal{H}$;

$$\begin{aligned} &\text{Case IIa} \quad \text{iff} \quad (1 + H_0)^{-1/2} h \in \mathrm{Dom}(1 + H_0)^{1/2} H_0^{-1/2}; \\ &\text{Case IIb} \quad \text{iff} \quad (1 + H_0)^{-1/2} h \notin \mathrm{Dom}(1 + H_0)^{1/2} H_0^{-1/2}. \end{aligned}$$

Case II iff $h \in (1 + H_0)\mathcal{H}$, $h \notin (1 + H_0)^{1/2}\mathcal{H}$;

$$\begin{aligned} &\text{Case IIIa} \quad \text{iff} \quad (1 + H_0)^{-1} h \in \mathrm{Dom}(1 + H_0)^{1/2} H_0^{-1/2}; \\ &\text{Case IIIb} \quad \text{iff} \quad (1 + H_0)^{-1} h \notin \mathrm{Dom}(1 + H_0)^{1/2} H_0^{-1/2}. \end{aligned}$$

In Case Ia and IIa we set

$$\lambda_{\mathrm{Kr}} := -(h|H_0^{-1}h)^{-1}. \tag{6.4.14}$$

Note that λ_{Kr} is negative. (In all other cases one could interpret $(h|H_0^{-1}h)$ as $+\infty$, and therefore one can then set $\lambda_{\mathrm{Kr}} := 0$). We have

$$\lim_{x\to-\infty} g(x) = -\lambda^{-1}, \quad g(0) = -\lambda^{-1} + \lambda_{\mathrm{Kr}}^{-1}.$$

Therefore, H_λ is positive for $\lambda_{\mathrm{Kr}} \leq \lambda \leq \infty$. For $\lambda < \lambda_{\mathrm{Kr}}$, H_λ has a single negative eigenvalue β, which is the solution of

$$\lambda(h|(H_0 - \beta)^{-1}h) = -1. \tag{6.4.15}$$

In Case IIa $H_{\lambda_{\mathrm{Kr}}}$ is the Krein extension of $H_{\min}$ and H_∞ is the Friedrichs extension.

In Case Ib and IIb we have

$$\lim_{x\to-\infty} g(x) = -\lambda^{-1}, \quad g(0) = -\infty.$$

H_λ is positive for $0 \leq \lambda \leq \infty$. For $\lambda < 0$, H_λ has a single negative negative eigenvalue β, which is the solution of (6.4.15). In Case IIb H_0 is the Krein extension of $H_{\min}$ and H_∞ is its

Friedrichs extension.

In Case III we will use several kinds of parameters, always putting them in brackets. In particular, it is natural to rename H_0 and call it $H_{(\infty)}$. It is the Friedrichs extension of $H_{\min}$.

In Case IIIa we have

$$\lim_{x\to-\infty} g(x) = \infty, \quad g(0) =: \gamma_0,$$

where γ_0 is a finite real number that can be used to parametrize H, so that

$$g(z) \;=\; \gamma_0 - \big(h|(H_0 - z)^{-1}H_0^{-1}h\big)\, z.$$

$H_{(\gamma_0)}$ is an increasing function of γ_0. It is positive for $0 \leq \gamma_0$. It has a single negative eigenvalue at β solving

$$\gamma_0 = (h|(H_0 - \beta)^{-1}H_0^{-1}h)\beta.$$

The Krein extension corresponds to $\gamma_0 = 0$.

In Case IIIb

$$\lim_{x\to-\infty} g(x) = \infty, \quad g(0) = -\infty.$$

A natural way to parametrize the Hamiltonian is by $g(z_0)$ for some fixed $z_0 \in]-\infty, 0[$, say $\gamma_{-1} := g(-1)$. This yields

$$g(z) \;=\; \gamma_{-1} - \big(h|(H_0 - z)^{-1}(H_0 + 1)^{-1}h\big)\,(z+1).$$

H is an increasing function of γ_{-1} on $\mathbb{R} \cup \{\infty\}$. The Krein extension is $H_{(\infty)}$ (and coincides with the Friedrichs extension).

$H_{(\gamma_{-1})}$ has a single negative eigenvalue β for all $\gamma_{-1} \in \mathbb{R}$. β is an increasing function of γ_{-1}.

If we use the cut-off method in Case III, then $\lambda_\Lambda \nearrow 0$. Thus we should think of λ as infinitesimally small negative.

Chapter 7

Friedrichs Hamiltonians and their renormalization

7.1 Construction

Let H_0 be again a self-adjoint operator on the Hilbert space $\mathcal{H}$. Let $\epsilon \in \mathbb{R}$ and $h \in \mathcal{H}$. The following operator on the Hilbert space $\mathbb{C} \oplus \mathcal{H}$ is often called the Friedrichs Hamiltonian:

$$G := \begin{bmatrix} \epsilon & (h| \\ |h) & H_0 \end{bmatrix}. \tag{7.1.1}$$

We would like to describe how to define the Friedrichs Hamiltonian if h is not necessarily a bounded functional on $\mathcal{H}$. It will turn out that it is natural to consider 3 types of h:

$$\mathrm{I.}\ h \in \mathcal{H}, \qquad \mathrm{II.}\ h \in \langle H_0\rangle^{1/2}\mathcal{H} \setminus \mathcal{H}, \qquad \mathrm{III.}\ h \in \langle H_0\rangle\mathcal{H} \setminus \langle H_0\rangle^{1/2}\mathcal{H}, \tag{7.1.2}$$

Clearly, in the case I G is self-adjoint on $\mathbb{C} \oplus \mathrm{Dom}\, H_0$. We will see that in the case II one can easily define G as a self-adjoint operator, but its domain is no longer $\mathbb{C} \oplus \mathrm{Dom}\, H_0$. In the case III, strictly speaking, the formula (7.1.1) does not make sense. Nevertheless, it is possible to define a renormalized Friedrichs Hamiltonian. To do this one needs to renormalize the parameter ϵ. This procedure resembles the renormalization of mass in quantum field theory.

Let us first consider the case $h \in \mathcal{H}$. As we said earlier, the operator G with $\mathrm{Dom}\, G = \mathbb{C} \oplus \mathrm{Dom}\, H_0$ is self-adjoint. It is well known that the resolvent of G can be computed exactly. In fact, for $z \notin \mathrm{sp} H_0$ define the analytic function

$$f(z) := \epsilon + (h|(z - H_0)^{-1}h). \tag{7.1.3}$$

Then for $z \in \mathbb{C}\backslash \mathrm{sp} H_0$, $f(z) \neq z$ the resolvent $Q(z) := (z-G)^{-1}$ is given by

$$\begin{aligned} Q(z) \ &= \ \begin{bmatrix} 0 & 0 \\ 0 & (z-H_0)^{-1} \end{bmatrix} \\ &+ \Big(z - f(z)\Big)^{-1} \begin{bmatrix} \mathbb{1} & (h|(z-H_0)^{-1} \\ (z-H_0)^{-1}|h) & (z-H_0)^{-1}|h)(h|(z-H_0)^{-1} \end{bmatrix}. \end{aligned} \tag{7.1.4}$$

Theorem 7.1.1 *Assume that:*

(A) $h \in \langle H_0\rangle^{1/2}\mathcal{H}$, $\epsilon \in \mathbb{R}$. *Let* $Q(z)$ *be given by (7.1.4) with* $f(z)$ *defined by (7.1.3), or*

(B) $h \in \langle H_0\rangle\mathcal{H}$, $\gamma \in \mathbb{R}$. *Let* $Q(z)$ *be given by (7.1.4) with* $f(z)$ *defined by*

$$\begin{cases} \partial_z f(z) = -\left(h|(z-H_0)^{-2}h\right), \\ \frac{1}{2}\big(f(\mathrm{i}) + f(-\mathrm{i})\big) = \gamma. \end{cases} \tag{7.1.5}$$

Then for all $z \in \mathbb{C}\backslash \mathrm{sp} H_0$, $f(z) \neq z$ *:*

(1) $Q(z)$ *is a pseudoresolvent;*

(2) $\mathrm{Ker} Q(z) = \{0\}$;

(3) $\mathrm{Ran}\, Q(z)$ *is dense in* $\mathbb{C} \oplus \mathcal{H}$;

(4) $Q(z)^* = Q(\overline{z})$.

Therefore, there exists a unique densely defined self-adjoint operator G *such that* $Q(z) = (z - G)^{-1}$.

Proof. Let $z \in \mathbb{C} \backslash \mathrm{sp} H_0$, $f(z) \neq z$. It is obvious that $Q(z)$ is bounded and satisfies (4). We easily see that both in the case (A) and (B) the function $f(z)$ satisfies

$$f(z_1) - f(z_2) = -(z_1 - z_2)(h|(z_1 - H_0)^{-1}(z_2 - H_0)^{-1}|h). \tag{7.1.6}$$

Direct computations using (7.1.6) show the first resolvent formula.

Let $(\alpha, f) \in \mathbb{C} \oplus \mathcal{H}$ be such that $(\alpha, f) \in \mathrm{Ker} Q(z)$. Then

$$0 = (z - f(z))^{-1}\Big(\alpha + (h|(z - H_0)^{-1} f)\Big), \tag{7.1.7}$$

$$0 = (z - H_0)^{-1} f + (z - H_0)^{-1} h (z - f(z))^{-1}\Big(\alpha + (h|(z - H_0)^{-1} f)\Big). \tag{7.1.8}$$

Inserting (7.1.7) into (7.1.8) we get $0 = (z - H_0)^{-1} f$ and hence $f = 0$. Now (7.1.7) implies $\alpha = 0$, so $\mathrm{Ker} Q(z) = \{0\}$.

Using (2) and (4) we get $(\mathrm{Ran}\, Q(z))^\perp = \mathrm{Ker} Q(z)^* = \mathrm{Ker} Q(\overline{z}) = \{0\}$. Hence (3) holds. □

It is easy to solve (7.1.5):

$$\begin{aligned} f(z) := \;& \gamma + \big(h|((z-H_0)^{-1} + H_0(1+H_0^2)^{-1})h\big) \\ = \;& \gamma + \Big(h|(\tfrac{\mathrm{i}-z}{2(z-H_0)(\mathrm{i}-H_0)} - \tfrac{\mathrm{i}+z}{2(z-H_0)(-\mathrm{i}-H_0)})h\Big) \end{aligned} \tag{7.1.9}$$

7.2 The cut-off method

Let $h \in \langle H_0 \rangle \mathcal{H}$ and $\gamma \in \mathbb{R}$. We can also use the cut-off method. For all $\Lambda > 0$ we define h_Λ as in (6.2.10), that is $h_\Lambda := 1\!\!1_{[-\Lambda,\Lambda]}(H_0)\, h,$. We set

$$\epsilon_\Lambda := \gamma + (h_\Lambda | H_0(1+H_0^2)^{-1} h_\Lambda).$$

For all $\Lambda > 0$, the cut-off Friedrichs Hamiltonian

$$G_\Lambda := \begin{bmatrix} \epsilon_\Lambda & (h_\Lambda| \\ |h_\Lambda) & H_0 \end{bmatrix}$$

is well defined and we can compute its resolvent, $Q_\Lambda(z) := (z - G_\Lambda)^{-1}$:

$$\begin{aligned} Q_\Lambda(z) \;&=\; \begin{bmatrix} 0 & 0 \\ 0 & (z-H_0)^{-1} \end{bmatrix} \qquad (7.2.10) \\ &+ \Big(z - f_\Lambda(z)\Big)^{-1} \begin{bmatrix} 1 & (h_\Lambda|(z-H_0)^{-1} \\ (z-H_0)^{-1}|h_\Lambda) & (z-H_0)^{-1}|h_\Lambda)(h_\Lambda|(z-H_0)^{-1} \end{bmatrix}. \end{aligned}$$

where

$$f_\Lambda(z) := \epsilon_\Lambda + (h_\Lambda|(z-H_0)^{-1}h_\Lambda). \qquad (7.2.11)$$

Note that ϵ_Λ is chosen such a way that the following renormalization condition is satisfied: $\frac{1}{2}\left(f_\Lambda(\mathrm{i}) + f_\Lambda(-\mathrm{i})\right) = \gamma$.

Theorem 7.2.1 *Assume that $h \in \langle H_0\rangle\mathcal{H}$. Then $\lim\limits_{k\to\infty} Q_\Lambda(z) = Q(z)$, where $Q(z)$ is given by (7.1.4) and $f(z)$ is given by (7.1.9). If H_0 is bounded from below, then $\lim\limits_{k\to\infty} \epsilon_\Lambda = \infty$.*

Proof. The proof is obvious if we note that $\lim\limits_{k\to\infty} \|(z-H_0)^{-1}h - (z-H_0)^{-1}h_\Lambda\| = 0$ and $\lim\limits_{k\to\infty} f_\Lambda(z) = f(z)$. $\square$

Thus the cut-off Friedrichs Hamiltonian is norm resolvent convergent to the renormalized Friedrichs Hamiltonian.

7.3 Eigenvectors and resonances

Let $\beta \notin \mathrm{sp}H_0$, If $\beta = f(\beta) = 0$ then G has an eigenvalue at β. The corresponding eigenprojection equals

$$\mathbb{1}_\beta(G) = (1 + (h|(\beta - H_0)^{-2}|h))^{-1} \begin{bmatrix} 1 & (h|(\beta-H_0)^{-1} \\ (\beta-H_0)^{-1}|h) & (\beta-H_0)^{-1}|h)(h|(\beta-H_0)^{-1} \end{bmatrix}.$$

It may happen that $\mathbb{C}\backslash\mathrm{sp}H_0 \ni z \mapsto f(z)$ extends to an analytic multivalued function accross some parts of $\mathrm{sp}H_0$. Then so does the resolvent $(z-G)^{-1}$ sandwiched between a certain class of vectors, in particular, between

$$w := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{7.3.12}$$

$$\big(w\big|(z-G)^{-1}w\big) = (z - f(z))^{-1}.$$

It may happen that we obtain a solution of

$$f(\beta) = \beta$$

in this non-physical sheet of the complex plane. This gives a pole of the resolvent called a resonance.

Suppose that we replace h with λh and ϵ with $\epsilon_0 + \lambda^2\alpha$ and assume that we have Case I or II with λ small.

Then if $\epsilon_0 \notin \mathrm{sp} H_0$, we have an approximate expression for the eigenvalue for small λ:

$$\epsilon_\lambda = \epsilon_0 + \lambda^2\alpha + \lambda^2(h|(\epsilon_0 - H_0)^{-1}h) + O(\lambda^4).$$

If $\epsilon_0 \in \mathrm{sp} H_0$, then the eigenvalue typically disappears and we obtain an approximate formula for the resonance:

$$\begin{aligned}\epsilon_\lambda &= \epsilon_0 + \lambda^2\alpha + \lambda^2(h|(\epsilon_0 + \mathrm{i}0 - H_0)^{-1}h) + O(\lambda^4) \\ &= \epsilon_0 + \lambda^2\alpha + \lambda^2(h|\mathcal{P}(\epsilon_0 - H_0)^{-1}h) - \lambda^2\mathrm{i}\pi(h|\delta(H_0)h) + O(\lambda^4).\end{aligned}$$

Suppose now that $\epsilon_0 = 0$. Then we have the weak coupling limit:

$$\lim_{\lambda\searrow 0}(w|\mathrm{e}^{-\mathrm{i}\frac{t}{\lambda^2}G_\lambda}w) = \exp\left(-\mathrm{i}t\alpha + \mathrm{i}t(h|\mathcal{P}(H_0^{-1})h) - t\pi(h|\delta(H_0)h)\right).$$

7.4 Dissipative semigroup from a Friedrichs Hamiltonian

Consider $L^2(\mathbb{R})$, $\epsilon \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and

$$H_0 v(k) := kv(k), \quad v \in L^2(\mathbb{R}), \quad k \in \mathbb{R}.$$

Then $\mathbb{R} \ni k \mapsto 1(k) = 1$ does not belong to $\langle H_0 \rangle^{1/2} L^2(\mathbb{R})$, however it belongs to $\langle H_0 \rangle L^2(\mathbb{R})$. We will see that

$$G = \begin{bmatrix} \epsilon & \lambda(1| \\ \overline{\lambda}|1) & H_0 \end{bmatrix} \tag{7.4.13}$$

is a well defined Friedrichs Hamiltonian without renormalizing λ, even though it is only type III.

Set $1_\Lambda(k) := 1\!\!1_{[-\Lambda,\Lambda]}(k)$. We approximate (7.4.13) by

$$G_\Lambda = \begin{bmatrix} \epsilon & \lambda(1_\Lambda| \\ \overline{\lambda}|1_\Lambda) & H_0 \end{bmatrix} \tag{7.4.14}$$

Note that (7.4.14) has a norm resolvent limit, which can be denoted (7.4.13). In fact,

$$f(z) = \epsilon + \lim_{\Lambda\to\infty} \int_{\Lambda}^{-\Lambda} \frac{|\lambda|^2}{z-k} \mathrm{d}k = \begin{cases} \epsilon - \mathrm{i}\pi|\lambda|^2 & \mathrm{Im} z > 0, \\ \epsilon + \mathrm{i}\pi|\lambda|^2 & \mathrm{Im} z < 0. \end{cases}$$

If w is the distinguished vector (7.3.12), then

$$\begin{aligned} \big(w|(z-G)^{-1}w\big) &= (z-\epsilon \pm \mathrm{i}\pi|\lambda|^2)^{-1}, \quad \pm\mathrm{Im} z > 0, \\ \big(w|\mathrm{e}^{-\mathrm{i}tG}w\big) &= \mathrm{e}^{-\mathrm{i}\epsilon t - \pi|\lambda|^2|t|}. \end{aligned}$$

Chapter 8

Momentum in one dimension

8.1 Distributions on $\mathbb{R}$

The space of distributions on $\mathbb{R}$ is denoted $\mathcal{D}'(\mathbb{R})$. Note that $L^1_{\mathrm{loc}}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$.

Proposition 8.1.1 (1) *Let* $g \in L^1_{\mathrm{loc}}(\mathbb{R})$. *Then*

$$\int_0^x g(y)\mathrm{d}y =: f(x) \tag{8.1.1}$$

is a continuous function and $f' = g$, where we use the derivative in the distributional sense.

(2) *If $g \in L^p(\mathbb{R})$ with $1 \leq p$, then $g \in L^1_{\text{loc}}(\mathbb{R})$ and so $f(x)$ defined in (8.1.1) is a continuous function.*

(3) *If $f \in C^1(\mathbb{R})$, then f' in the classical and ditributional sense coincide.*

θ will denote the Heavyside function.

8.2 Momentum on the line

Consider the Hilbert space $L^2(\mathbb{R})$.

The equation

$$U(t)f(x) := f(x-t), \quad f \in L^2(\mathbb{R}), \;\; t \in \mathbb{R},$$

defines a unitary strongly continuous group.

The momentum operator p is defined by on the domain

$$\mathrm{Dom}\, p := \{f \in L^2(\mathbb{R}) \ : \ f' \in L^2(\mathbb{R})\}$$

by

$$pf(x) := \frac{1}{\mathrm{i}}\partial_x f(x), \quad f \in \mathrm{Dom}\, p. \tag{8.2.2}$$

Its graph scalar product is

$$(f|g)_p = \int_{-\infty}^{\infty} \big(\overline{f(x)}g(x) + \overline{f'(x)}g'(x)\big)\mathrm{d}x.$$

Theorem 8.2.1 (1) $U(t) = \mathrm{e}^{-\mathrm{i}tp}$.

(2) p *is a self-adjoint operator.*

(3) $C_\mathrm{c}^\infty(\mathbb{R})$ *is an essential domain of* p*.*

(4) $\mathrm{sp}\, p = \mathbb{R}$, $\mathrm{sp_p} p = \emptyset$.

(5) *The integral kernel of* $(z-p)^{-1}$ *equals*

$$R(z,x,y) = \begin{cases} -\mathrm{i}\theta(x-y)\mathrm{e}^{iz(x-y)}, & \mathrm{Im} z > 0, \\ +\mathrm{i}\theta(y-x)\mathrm{e}^{iz(x-y)}, & \mathrm{Im} z < 0. \end{cases}$$

Proof. (1): Let A be the generator, $f \in \mathrm{Dom}\, A$. Then for any $\phi \in C_{\mathrm{c}}^{\infty}(\mathbb{R})$

$$(\phi|Af) \to \frac{1}{t}(\phi|U(t)f - f) \;=\; \frac{1}{t}\int \big(\overline{\phi(x+t)} - \overline{\phi(x)}\big) f(x)\mathrm{d}x \to \int \overline{\phi'(x)} f(x)\mathrm{d}x.$$

Therefore, $Af = -f'$ in the distributional sense.

Let $f \in L^2$, $g := f' \in L^2$. Then $f \in C(\mathbb{R})$ and

$$\frac{1}{t}\big(f(x-t) - f(x) = \frac{1}{t}\int_{x-t}^{x} g(y)\mathrm{d}y = j_t * g \to g, \tag{8.2.3}$$

where we $j_t := \begin{cases} 1/t, & y \in [-t,0], \\ 0 & y \notin [-t,0]. \end{cases}$ and (8.2.3) is understood in the L^2 sense. Therefore, $f \in \mathrm{Dom}\, p$.

(3): $C_{\mathrm{c}}^{\infty}(\mathbb{R})$ is a dense subspace of $L^2(\mathbb{R})$ left invariant by $U(t)$. Therefore, it is an essential domain.

(5): For $\mathrm{Im} z > 0$

$$(z-p)^{-1} = -\mathrm{i}\int_0^{\infty} \mathrm{e}^{\mathrm{i}zt} U(t)\mathrm{d}t.$$

Hence

$$(z-p)^{-1} f(x) = -\mathrm{i}\int_0^{\infty} \mathrm{e}^{\mathrm{i}zt} f(x-t)\mathrm{d}t = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(x-y)z} \theta(x-y) f(y)\mathrm{d}y.$$

For $\mathrm{Im} z < 0$ we can use

$$(z-p)^{-1*} = (\overline{z}-p)^{-1}.$$

(4): Let $k \in \mathbb{R}$. Consider $f_{\epsilon,k} = \sqrt{\pi\epsilon}\mathrm{e}^{-\epsilon x^2 + \mathrm{i}kx}$. Then $\|f_{\epsilon,k}\| = 1$, $f_{\epsilon,k} \in \mathrm{Dom}\, p$ and $(k-p)f_{\epsilon,k} \to 0$ as $\epsilon \to 0$. Hence $k \in \mathrm{sp} p$.

Suppose that $f \in \mathrm{Dom}\, p$ and $pf = kf$. Clearly, $f \in \mathrm{Dom}\, p^2$. Hence, by Theorem 9.1.1, $f \in C^1(\mathbb{R})$ and $pf = -\mathrm{i}\partial_x f = kf$. It is well known that the only solution is $f = c\mathrm{e}^{\mathrm{i}kx}$, which does not belong to $L^2(\mathbb{R})$. $\square$

Proposition 8.2.2 (1) $\mathrm{Dom}\, p \subset C_\infty(\mathbb{R})$ *and* $\mathrm{Dom}\, p \ni f \mapsto f(x) \in \mathbb{C}$ *is a continuous functional.*

(2) *If* $f \in \mathrm{Dom}\, p$ *and* $pf \in C(\mathbb{R})$, *then* $f \in C^1(\mathbb{R})$ *and (8.2.2) is true in the classical sense.*

(3) *If* $f \in \mathrm{Dom}\, p$ *and* $f = 0$ *on* $]a,b[$, *then* $pf = 0$ *on* $]a,b[$.

Proof. (1): $\mathrm{Dom}\, p = \mathrm{Ran}\,(\mathrm{i} - p)^{-1}$. Now $(\mathrm{i} - p)^{-1}$ is the convolution with $-\mathrm{i}\theta(x)\mathrm{e}^{-|x|}$, which belongs to $L^2(\mathbb{R})$. The convolution of two $L^2(\mathbb{R})$ functions belongs to $C_\infty(\mathbb{R})$.

(2) Let $f \in \mathrm{Dom}\, p$, $g \in C(\mathbb{R})$ and $pf = g$. Let $x \in \mathbb{R}$, $r > 0$. Set $h := 1_{[x,x+r]}$. Then

$$\begin{aligned} t^{-1}(h|U(t)f - f) &= t^{-1} \int_{x-t}^{x+r-t} f(y)\mathrm{d}y - t^{-1} \int_x^{x+r} f(y)\mathrm{d}y \\ &= -t^{-1} \int_{x+r-t}^{x+r} f(y)\mathrm{d}y + t^{-1} \int_{x-t}^{x} f(y)\mathrm{d}y \to -f(x+r) + f(x). \end{aligned}$$

where we used the continuity of f. Therefore

$$\mathrm{i}(h|g) = \mathrm{i} \int_x^{x+r} g(y)\mathrm{d}y = -f(x+r) + f(x).$$

Hence, using the continuity of g,

$$\lim_{r\to 0}\frac{f(x+r)-f(x)}{r}=-\mathrm{i}g(x).$$

(3) is obvious for $f\in C_{\mathrm{c}}^1(\mathbb{R})$. It extends by density. □

Proposition 8.2.3 (1) *The spaces*

$$\{f\in \operatorname{Dom} p\ :\ f(x)=0,\ x<0\},\tag{8.2.4}$$

$$\{f\in \operatorname{Dom} p\ :\ f(x)=0,\ x>0\}.\tag{8.2.5}$$

are mutually orthogonal in $\operatorname{Dom} p$.

(2) *The orthogonal complement of the direct sum of (8.2.4) and (8.2.5) is spanned by* $\mathrm{e}^{-|x|}$.

Proof. (2): We easily check the orthogonality of $\mathrm{e}^{-|x|}$ to (8.2.4) and (8.2.5).

Let $f\in L_1^2(\mathbb{R})$. Set $f_\pm(x):=\theta(\pm x)\big(f(x)-f(0)\mathrm{e}^{-|x|}\big)$. Then

$$f(x)=f(0)\mathrm{e}^{-|x|}+f_-(x)+f_+(x).$$

□

8.3 Momentum on the half-line

Consider the Hilbert space $L^2([0,\infty[)$.

Define the semigroups

$$U_{\leftarrow}(t)f(x) := f(x+t), \quad t \geq 0.$$

$$U_{\rightarrow}(t)f(x) := \begin{cases} f(x-t), & x \geq t \geq 0. \\ 0, & t > x, \end{cases}$$

Define $p_{\max}$ by

$$p_{\max}f(x) := \frac{1}{\mathrm{i}}\partial_x f(x), \quad f \in \operatorname{Dom} p_{\max} := \{\mathbb{1}_{[0,\infty[}f \ : \ f \in \operatorname{Dom} p\}. \tag{8.3.6}$$

Note that the graph scalar product of $p_{\max}$ is

$$(f|g)_{p_{\max}} = \int_0^\infty \left(\overline{f(x)}g(x) + \overline{f'(x)}g'(x)\right)\mathrm{d}x.$$

Define the operator $p_{\min}$ as the restriction of $p_{\max}$ to the domain

$$\mathrm{Dom}\, p_{\min} := \{f \in \mathrm{Dom}\, p \ :\ f(x) = 0,\ x < 0\}.$$

(In the definitions of $\mathrm{Dom}\, p_{\max}$ and $\mathrm{Dom}\, p_{\min}$ we used concepts defined in the space $L^2(\mathbb{R})$, however it is easy to see that both are subspaces of $L^2([0,\infty[)$.)

Theorem 8.3.1 (1) *We have* $U_{\leftarrow}(t) = \mathrm{e}^{\mathrm{i}tp_{\max}}$ *and* $U_{\rightarrow}(t) = \mathrm{e}^{-\mathrm{i}tp_{\min}}$.

(2) $p_{\min} \subset p_{\max}$, $p_{\min}^* = p_{\max}$, $p_{\max}^* = p_{\min}$; *the operators* $p_{\min}$ *and* $-p_{\max}$ *are m-dissipative (in particular, they are closed); the operator* $p_{\min}$ *is hermitian.*

(3) $\mathrm{Dom}\, p_{\min}$ *is a subspace of* $\mathrm{Dom}\, p_{\max}$ *of codimension 1 and its orthogonal complement is spanned by* $\mathbb{1}_{[0,\infty[}(x)\mathrm{e}^{-x}$.

(4) $C_{\mathrm{c}}^\infty([0,\infty[)$ *is an essential domain of* $p_{\max}$ *and* $C_{\mathrm{c}}^\infty(]0,\infty[)$ *is an essential domain of* $p_{\min}$.

(5) $\mathrm{sp} p_{\max} = \mathrm{sp}_{\mathrm{p}} p_{\max} = \{\mathrm{Im} z \geq 0\}$, $\mathrm{sp} p_{\min} = \{\mathrm{Im} z \leq 0\}$, $\mathrm{sp}_{\mathrm{p}} p_{\min} = \emptyset$,

$$p_{\max}\mathrm{e}^{\mathrm{i}zx} = z\mathrm{e}^{\mathrm{i}zx}, \quad \mathrm{e}^{\mathrm{i}zx} \in \mathrm{Dom}\, p_{\max}, \quad \mathrm{Im} z > 0; \tag{8.3.7}$$

(6) *The integral kernels of* $(z-p_{\max})^{-1}$ *and* $(z-p_{\min})^{-1}$ *are equal*

$$R_{\max}(z,x,y) = \mathrm{i}\theta(y-x)\mathrm{e}^{\mathrm{i}z(x-y)},\ \mathrm{Im}z<0.$$

$$R_{\min}(z,x,y) = -\mathrm{i}\theta(x-y)\mathrm{e}^{\mathrm{i}z(x-y)},\ \mathrm{Im}z>0.$$

8.4 Momentum on an interval I

Consider the Hilbert space $L^2([-\pi,\pi])$.

Define $p_{\max}$ as an operator with domain

$$\mathrm{Dom}\, p_{\max} =:= \{\mathbb{1}_{[-\pi,\pi]}f \ : \ f\in\mathrm{Dom}\, p\}$$

and

$$p_{\max}f(x) := \frac{1}{\mathrm{i}}\partial_x f(x), \quad f\in\mathrm{Dom}\, p_{\max}. \tag{8.4.8}$$

Note that the graph scalar product for $p_{\max}$ is

$$(f|g)_{p_{\max}} = \int_{-\pi}^{\pi} \big(\overline{f(x)}g(x) + \overline{f'(x)}g'(x)\big)\mathrm{d}x, \quad f,g \in \mathrm{Dom}\, p_{\max}.$$

Define the operator $p_{\min}$ as the restriction of $p_{\max}$ to the domain

$$\mathrm{Dom}\, p_{\min} := \{f \in \mathrm{Dom}\, p \ : \ f(x) = 0,\ x \notin]-\pi,\pi[\}.$$

Theorem 8.4.1 (1) *Neither $p_{\max}$ nor $p_{\min}$ generate a semigroup.*

(2) $p_{\min} \subset p_{\max}$, $p_{\min}^* = p_{\max}$, $p_{\max}^* = p_{\min}$*; the operators $p_{\min}$ and $p_{\max}$ are closed; the operator $p_{\min}$ is hermitian.*

(3) $C^\infty([-\pi,\pi])$ *is an essential domain of $p_{\max}$ and $C_{\mathrm{c}}^\infty(]-\pi,\pi[)$ is an essential domain of $p_{\min}$.*

(4) $\mathrm{sp} p_{\max} = \mathrm{sp}_{\mathrm{p}} p_{\max} = \mathbb{C}$, $\mathrm{sp} p_{\min} = \mathbb{C}$, $\mathrm{sp}_{\mathrm{p}} p_{\min} = \emptyset$,

$$p_{\max}\mathrm{e}^{\mathrm{i}zx} = z\mathrm{e}^{\mathrm{i}zx}, \qquad z \in \mathbb{C}, \tag{8.4.9}$$

8.5 Momentum on an interval II

Let $\kappa \in \mathbb{C}$. Define the family of groups on $L^2([-\pi, \pi])$ by

$$U_\kappa(t)\phi(x) = \mathrm{e}^{\mathrm{i}2\pi n\kappa}\phi(x-t), \quad -(2n-1)\pi < x - t < -(2n+1)\pi, \ n \in \mathbb{Z}.$$

Let the operator p_κ be defined as the restriction of $p_{\max}$ to

$$\mathrm{Dom}\, p_\kappa = \{f \in \mathrm{Dom}\, p_{\max} \ : \ \mathrm{e}^{\mathrm{i}2\pi\kappa} f(-\pi) = f(\pi)\}.$$

Theorem 8.5.1 (1) $U_\kappa(t) = \mathrm{e}^{-\mathrm{i}tp_\kappa}$.

(2) $\|U_\kappa(t)\| = \mathrm{e}^{2\pi n \mathrm{Im}\kappa}$, $2\pi(n-1) < t \leq 2\pi n$, $n \in \mathbb{Z}$.

(3) *The semigroup* $[0, \infty[\ni t \mapsto U_\kappa(t)$ *is of type* $(1, 0)$ *for* $\mathrm{Im}\kappa \leq 0$ *and of type* $(\mathrm{e}^{2\pi \mathrm{Im}\kappa}, \mathrm{Im}\kappa)$ *for* $\mathrm{Im}\kappa \geq 0$.

(4) $p_\kappa^* = p_{\overline{\kappa}}, \quad p_\kappa = p_{\kappa+1}; \quad p_{\min} \subset p_\kappa \subset p_{\max}$. *Operators* p_κ *are closed. For* $\kappa \in \mathbb{R}$ *they are self-adjoint.*

(5) $\{f \in C^\infty([-\pi, \pi]) \ : \ \mathrm{e}^{\mathrm{i}2\pi\kappa} f(-\pi) = f(\pi)\}$ *is an essential domain of* p_κ.

(6) $\mathrm{sp} p_\kappa = \mathrm{sp_p} p_\kappa = \mathbb{Z} + \kappa$,

$$p_\kappa \mathrm{e}^{\mathrm{i}(n+\kappa)x} = (n+\kappa)\mathrm{e}^{\mathrm{i}(n+\kappa)x}, \;\; n \in \mathbb{Z}.$$

(7) *The integral kernel of* $(z - p_\kappa)^{-1}$ *equals*

$$R_\kappa(z,x,y) = \frac{1}{2\sin\pi(z-\kappa)}\left(\mathrm{e}^{-\mathrm{i}(z-\kappa)\pi}\mathrm{e}^{\mathrm{i}z(x-y)}\theta(x-y) + \mathrm{e}^{\mathrm{i}(z-\kappa)\pi}\mathrm{e}^{\mathrm{i}z(x-y)}\theta(y-x)\right).$$

(8) *The operators* p_κ *are similar to one another up to an additive constant:*

$$\mathrm{Dom}\, p_\kappa = \mathrm{e}^{\mathrm{i}\kappa x}\,\mathrm{Dom}\, p_0, \qquad p_\kappa = \mathrm{e}^{\mathrm{i}\kappa x} p_0 \mathrm{e}^{-\mathrm{i}\kappa x} + \kappa. \tag{8.5.10}$$

8.6 Momentum on an interval III

Define the contractive semigroups on $L^2([-\pi,\pi])$:

$$U_\leftarrow(t)f(x) := \begin{cases} f(x+t), & |x+t| \le \pi, \\ 0 & |x+t| > \pi. \end{cases}$$

$$U_{\to}(t)f(x) := \begin{cases} f(x-t), & |x-t| \leq \pi, \\ 0 & |x-t| > \pi. \end{cases} .$$

Let the operator $p_{\pm \mathrm{i}\infty}$ be defined as the restriction of $p_{\max}$ to

$$\mathrm{Dom}\, p_{\pm \mathrm{i}\infty} = \{f \in \mathrm{Dom}\, p_{\max} \ : \ f(\pm\pi) = 0\}.$$

Theorem 8.6.1 (1) $U_{\leftarrow}(t) = \mathrm{e}^{\mathrm{i}tp_{+\mathrm{i}\infty}}$ *and* $U_{\to}(t) = \mathrm{e}^{-\mathrm{i}tp_{-\mathrm{i}\infty}}$*.*

(2) $p_{\pm\mathrm{i}\infty}^* = p_{\mp\mathrm{i}\infty}$*;* $p_{\min} \subset p_{\pm\mathrm{i}\infty} \subset p_{\max}$*. Operators* $p_{\pm\mathrm{i}\infty}$ *are closed.*

(3) $\mathrm{sp} p_{\pm\mathrm{i}\infty} = \emptyset$*.*

(4) *The integral kernel of* $(z - p_{\pm\mathrm{i}\infty})^{-1}$ *equals*

$$R_{\pm\mathrm{i}\infty}(z, x, y) = \pm\mathrm{i}\mathrm{e}^{\mathrm{i}z(x-y\pm\pi)}\theta(\pm y \mp x), \quad z \in \mathbb{C}.$$

Chapter 9

Laplacian

9.1 Sobolev spaces in one dimension

For $\alpha \in \mathbb{R}$ let $\langle p\rangle^{-\alpha} L^2(\mathbb{R})$ be the scale of Hilbert spaces associated with the operator p. It is called the scale of Sobolev spaces. We will focus in the case $\alpha \in \mathbb{N}$.

Theorem 9.1.1 (1)

$$\langle p\rangle^{-n} L^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \ : \ f^{(n)} \in L^2(\mathbb{R})\}.$$

(2) $\langle p\rangle^{-n}L^2(\mathbb{R}) \subset C^{n-1}(\mathbb{R})$ *and* $\langle p\rangle^{-n}L^2(\mathbb{R}) \ni f \mapsto f^{(j)}(x)$ *for* $j = 0,\dots,n-1$ *are continuous functionals depending continuously on* $x \in \mathbb{R}$.

Proof. We use induction. The step $n = 1$ was proven before.

Suppose that we know that $\langle p\rangle^{-n}L^2(\mathbb{R}) \subset C^n(\mathbb{R})$. Let $f \in \langle p\rangle^{-(n+1)}L^2(\mathbb{R})$. Then $(\mathrm{i}-p)f = g \in \langle p\rangle^{-n}L^2(\mathbb{R})$. Clearly, $\langle p\rangle^{-n-1}L^2(\mathbb{R}) \subset \langle p\rangle^{-n}L^2(\mathbb{R})$, hence $f \in C^{n-1}(\mathbb{R})$. Likewise, $g \in C^{n-1}(\mathbb{R})$, by the induction assumption. Now $pf = -g + \mathrm{i}f \in C^{n-1}(\mathbb{R})$. Hence, by Prop. 8.2.2 (2) $f \in C^n(\mathbb{R})$. $\square$

9.2 Laplacian on the line

Define the form $\mathfrak{d}$ by

$$\mathfrak{d}(f,g) := \int \overline{f'(x)}g'(x)\mathrm{d}x, \quad f,g \in \mathrm{Dom}\,\mathfrak{d} := \langle p\rangle^{-1}L^2(\mathbb{R}).$$

The operator p^2 on $L^2(\mathbb{R})$ will be denoted $-\Delta$. Thus

$$-\Delta f(x) = -\partial_x^2 f(x), \quad f \in \mathrm{Dom}(-\Delta) = \langle p\rangle^{-2} L^2(\mathbb{R}).$$

Theorem 9.2.1 (1) $-\Delta$ *is a positive self-adjoint operator.*

(2) $\mathrm{sp_p}(-\Delta) = \emptyset$.

(3) $\mathrm{sp}(-\Delta) = [0, \infty[$.

(4) *The integral kernel of* $(k^2 - \Delta)^{-1}$, *for* $\mathrm{Re}k > 0$, *is*

$$R(k, x, y) = \frac{1}{2k}\mathrm{e}^{-k|x-y|}.$$

(5) *The integral kernel of* $\mathrm{e}^{t\Delta}$ *is*

$$K(t, x, y) = (4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x-y)^2}{4t}}.$$

(6) *The form* $\mathfrak{d}$ *is closed and associated with the operator* $-\Delta$.

(7) $\{f \in C^2(\mathbb{R}) \cap L^2(\mathbb{R}) \ : \ f', f'' \in L^2(\mathbb{R})\}$ *is contained in* $\mathrm{Dom}(-\Delta)$ *and on this set*

$$-\Delta f(x) = -\partial_x^2 f(x).$$

(8) $C_{\mathrm{c}}^\infty(\mathbb{R})$ *is an essential domain of* $-\Delta$*.*

Proof. (4) Let $\mathrm{Re}k > 0$. Then

$$(\mathrm{i}k - p)^{-1}(x,y) = -\mathrm{i}\theta(x-y)\mathrm{e}^{-k|x-y|}, \quad (-\mathrm{i}k - p)^{-1}(x,y) = \mathrm{i}\theta(y-x)\mathrm{e}^{-k|x-y|}.$$

Now

$$\begin{aligned} (k^2 - \Delta)^{-1} &= (\mathrm{i}k - p)^{-1}(-\mathrm{i}k - p)^{-1} \\ &= (-2\mathrm{i}k)^{-1}\left((\mathrm{i}k - p)^{-1} - (-\mathrm{i}k - p)^{-1}\right). \end{aligned} \tag{9.2.1}$$

The integral kernel of (9.2.1) equals $(2k)^{-1}\mathrm{e}^{-k|x-y|}$.

(5) We have

$$\mathrm{e}^{t\Delta} = (2\pi\mathrm{i})^{-1}\int_\gamma (z-\Delta)^{-1}\mathrm{e}^{tz}\mathrm{d}z,$$

where γ is a contour of the form $\mathrm{e}^{-\mathrm{i}\alpha}]0,\infty[\cup\mathrm{e}^{\mathrm{i}\alpha}[0,\infty[$ bypassing 0, where $\pi/2<\alpha<\pi$. Hence

$$\mathrm{e}^{t\Delta}(x,y)=(2\pi\mathrm{i})^{-1}\int_{\tilde{\gamma}}\mathrm{e}^{-k|x-y|+tk^2}\mathrm{d}k$$

where $\tilde{\gamma}$ is a contour of the form $\mathrm{e}^{-\mathrm{i}\alpha/2}[0,\infty[\cup\mathrm{e}^{\mathrm{i}\alpha/2}[0,\infty[$. We put $k=\mathrm{i}u$ and obtain

$$\mathrm{e}^{t\Delta}(x,y)=(2\pi\mathrm{i})^{-1}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}u|x-y|-tu^2}\mathrm{i}\mathrm{d}u$$

□

9.3 Laplacian on the halfline I

Consider the space $L^2([0,\infty[)$. Define $-\Delta_{\max}$ by

$$-\Delta_{\max}f=-\partial_x^2 f,\quad f\in\mathrm{Dom}\big(-\Delta_{\max}\big):=\{\mathbb{1}_{[0,\infty[}f\ :\ f\in\langle p\rangle^{-2}L^2(\mathbb{R})\}.$$

Likewise, define $-\Delta_{\min}$ as the restriction of $-\Delta_{\max}$ to

$$\mathrm{Dom}\left(-\Delta_{\min}\right) := \{f \in \langle p\rangle^{-2}L^2(\mathbb{R}) \ : \ f(x)=0,\ x<0\}.$$

(Both $\mathrm{Dom}\left(-\Delta_{\max}\right)$ and $\mathrm{Dom}\left(-\Delta_{\min}\right)$ are defined using the space $L^2(\mathbb{R})$. It is easy to see that they are contained in $L^2([0,\infty[)$.)

Theorem 9.3.1 (1) $-\Delta_{\min}^* = -\Delta_{\max}$, $\quad -\Delta_{\min} \subset -\Delta_{\max}$.

(2) *The operators $-\Delta_{\min}$ and $-\Delta_{\max}$ are closed and $-\Delta_{\min}$ is hermitian.*

(3) $\mathrm{sp}_{\mathrm{p}}(-\Delta_{\max}) = \mathbb{C}\backslash[0,\infty[$, $\quad \mathrm{sp}_{\mathrm{p}}(-\Delta_{\min}) = \emptyset$

$$-\Delta_{\max}\mathrm{e}^{\mathrm{i}kx} = k^2\mathrm{e}^{\mathrm{i}kx},\ \mathrm{Im}k>0, \qquad \mathrm{e}^{\mathrm{i}kx} \in \mathrm{Dom}(-\Delta_{\max}).$$

(4) $\mathrm{sp}(-\Delta_{\max}) = \mathbb{C}$, $\ \mathrm{sp}(-\Delta_{\min}) = \mathbb{C}$.

(5) $-\Delta_{\min} = \left(p_{\min}\right)^2$, $\quad -\Delta_{\max} = \left(p_{\max}\right)^2$.

9.4 Laplacian on the halfline II

Let $\mu \in \mathbb{C} \cup \{\infty\}$. Let $-\Delta_\mu$ be the restriction of $-\Delta_{\max}$ to

$$\mathrm{Dom}(-\Delta_\mu) = \{f \in \mathrm{Dom}(-\Delta_{\max}) \quad : \quad \mu f(0) = f'(0)\}. \tag{9.4.2}$$

(If $\mu = \infty$, these are the Dirichlet boundary conditions, that means $f(0) = 0$, if $\mu = 0$, these are the Neumann boundary conditions, that means $f'(0) = 0$).

Define also the form $\mathfrak{d}_\mu$ as follows. If $\mu \in \mathbb{R}$, then

$$\mathfrak{d}_\mu(f,g) := \mu\overline{f(0)}g(0) + \int \overline{f'(x)}g'(x)\mathrm{d}x, \quad f,g \in \mathrm{Dom}\,\mathfrak{d}_\mu := \mathrm{Dom}\,p_{\max}.$$

For $\mu = \infty$,

$$\mathfrak{d}_\infty(f,g) := \int \overline{f'(x)}g'(x)\mathrm{d}x, \quad f,g \in \mathrm{Dom}\,\mathfrak{d}_\infty := \mathrm{Dom}\,p_{\min}.$$

Theorem 9.4.1 (1) $-\Delta_{\min} \subset -\Delta_\mu \subset -\Delta_{\max}$.

(2) $-\Delta_\mu^* = -\Delta_{\overline{\mu}}$.

(3) *The operator* $-\Delta_\mu$ *is a generator of a group. For* $\mu \in \mathbb{R} \cup \{\infty\}$ *it is self-adjoint.*

(4) $\mathrm{sp_p}(-\Delta_\mu) = \begin{cases} \{-\mu^2\}, & \mathrm{Re}\mu < 0; \\ \emptyset, & \text{otherwise;} \end{cases}$

$-\Delta_\mu \mathrm{e}^{\mu x} = -\mu^2 \mathrm{e}^{\mu x}, \ \mathrm{Re}\mu < 0, \qquad \mathrm{e}^{\mu x} \in \mathrm{Dom}(-\Delta_\mu).$

(5) $\mathrm{sp}(-\Delta_\mu) = \begin{cases} \{-\mu^2\} \cup [0, \infty[, & \mathrm{Re}\mu < 0, \\ [0, \infty[, & \text{otherwise.} \end{cases}$

(6) $-\Delta_0 = p_{\max}^* p_{\max}, \quad -\Delta_\infty = p_{\min}^* p_{\min}.$

(7) *The forms* $\mathfrak{d}_\mu$ *are closed and associated with the operator* $-\Delta_\mu$.

(8) *Let* $\mathrm{Re}k > 0$. *The integral kernel of* $(k^2 - \Delta_\mu)^{-1}$ *is equal*

$$R_\mu(k, x, y) = \frac{1}{2k}\mathrm{e}^{-k|x-y|} + \frac{1}{2k}\frac{(k-\mu)}{(k+\mu)}\mathrm{e}^{-k(x+y)},$$

in particular, for the Dirichlet boundary conditions,

$$R_\infty(z, x, y) = \frac{1}{2k}\mathrm{e}^{-k|x-y|} - \frac{1}{2k}\mathrm{e}^{-k(x+y)},$$

and for the Neumann boundary conditions

$$R_0(k,x,y) = \frac{1}{2k}\mathrm{e}^{-k|x-y|} + \frac{1}{2k}\mathrm{e}^{-k(x+y)}.$$

(9) *The semigroups* $\mathrm{e}^{t\Delta_\mu}$ *have the integral kernel*

$$K_\mu(t,x,y) = (4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x-y)^2}{4t}} + (2\pi)^{-1}\int_{-\infty}^{\infty}\frac{\mathrm{i}u-\mu}{\mathrm{i}u+\mu}\mathrm{e}^{-\mathrm{i}u(x+y)-tu^2}\mathrm{d}u,$$

In particular, in the Dirichlet case

$$K_\infty(t,x,y) = (4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x-y)^2}{4t}} - (4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x+y)^2}{4t}},$$

and in the Neumann case

$$K_0(t,x,y) = (4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x-y)^2}{4t}} + (4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x+y)^2}{4t}}.$$

The group $\mathrm{e}^{\mathrm{i}t\Delta_\mu}$ for $\mu \in \mathbb{R}\cup\{\infty\}$ describes a quantum particle with a potential well or bump at the end of the halfline.

The semigroup $\mathrm{e}^{t\Delta_\mu}$ for $\mu \in \mathbb{R}$ describes the diffusion with a sink or source at the end of the halfline. Note that $\mathrm{e}^{t\Delta_\mu}$ preserves the pointwise positivity. If $p_t = \mathrm{e}^{t\Delta_\mu} p_0$, $0 < a < b$, then

$$\partial_t \int_a^b p_t(x)\mathrm{d}x = p'(b) - p'(a).$$

$$\partial_t \int_0^a p_t(x)\mathrm{d}x = p'(a) - \mu p(0).$$

Thus at 0 there is a sink of p with the rate μ.

9.5 Neumann Laplacian on a halfline with the delta potential

On $L^2([0,\infty[)$ we define the cosine transform

$$U_{\mathrm{N}} f(k) := \pi^{-1/2} \int_0^\infty \cos kx f(x)\mathrm{d}x, \quad k \geq 0.$$

Note that U_{N} is unitary and $U_{\mathrm{N}}^2 = 1$.

Let Δ_{N} be the Laplacian on $L^2([0,\infty[)$ with the Neumann boundary condition. Clearly,

$$-U_{\mathrm{N}} \Delta_{\mathrm{N}} U_{\mathrm{N}}^* = k^2.$$

Let $|\delta)(\delta|$ be the quadratic form given by

$$(f_1|\delta)(\delta|f_2) = \overline{f_1(0)}f_2(0),$$

Note that it can be formally written as

$$\int_0^\infty \overline{f(x)}\delta(x)g(x)\mathrm{d}x,$$

and thus is interpreted as a "potential".

Let $(1|$ denote the functional on $L^2([0,\infty[)$ given by

$$(1|g) = \int_0^\infty g(k)\mathrm{d}k.$$

Using $\delta(x) = \pi^{-1}\int_0^\infty \cos kx \mathrm{d}x$ we deduce that

$$U_\mathrm{N}|\delta)(\delta|U_\mathrm{N}^* = \pi^{-1}|1)(1|.$$

Then

$$U_\mathrm{N}\left(-\Delta_\mathrm{N} + \lambda|\delta)(\delta|\right)U_\mathrm{N}^* = k^2 + \lambda\pi^{-1}|1)(1|$$

is an example of an Aronszajn-Donoghue Hamiltonian of type IIb, because

$$\int_0^\infty 1\mathrm{d}k = \infty, \quad \int_0^\infty \frac{1}{1+k^2}\mathrm{d}k < \infty, \quad \int_0^\infty \frac{1}{k^2}\mathrm{d}k = \infty.$$

9.6 Dirichlet Laplacian on a halfline with the δ' potential

On $L^2([0,\infty[)$ we define the sine transform

$$U_{\mathrm{D}} f(k) := \pi^{-1/2} \int_0^\infty \sin kx f(x)\mathrm{d}x, \quad k \geq 0.$$

Note that U_{D} is unitary and $U_{\mathrm{D}}^2 = 1$

Let Δ_{D} be the Laplacian on $L^2([0,\infty[)$ with the Dirichlet boundary condition. Clearly,

$$-U_{\mathrm{D}}\Delta_{\mathrm{D}}U_{\mathrm{D}}^* = k^2.$$

Using $-\delta'(x) = \pi^{-1}\int_0^\infty \sin kx\mathrm{d}x$ we deduce that

$$U_{\mathrm{D}}|\delta')(\delta'|U_{\mathrm{D}}^* = \pi^{-1}|k)(k|.$$

Here $|\delta')(\delta'|$ is the quadratic form given by

$$(f_1|\delta')(\delta'|f_2) = \overline{f_1'(0)}f_2'(0),$$

and $(k|$ is the functional on $L^2([0,\infty[)$ given by

$$(k|g) = \int_0^\infty kg(k)\mathrm{d}k.$$

Thus

$$U_\mathrm{D}\left(-\Delta_\mathrm{D} + \lambda|\delta')(\delta'|\right)U^* = k^2 + \lambda\pi^{-1}|k)(k|$$

is an example of an Aronszajn-Donoghue Hamiltonian of type IIIa, because

$$\int_0^\infty \frac{k^2}{1+k^2}\mathrm{d}k = \infty,\ \ \int_0^\infty \frac{k^2}{(1+k^2)^2}\mathrm{d}k < \infty,\ \ \int_0^\infty \frac{k^2}{(1+k^2)k^2}\mathrm{d}k < \infty.$$

9.7 Laplacian on $L^2(\mathbb{R}^d)$ with the delta potential

On $L^2(\mathbb{R}^d)$ we consider the unitary operator $U = (2\pi)^{d/2}\mathcal{F}$, where $\mathcal{F}$ is the Fourier transformation. Note that U is unitary.

Let Δ be the usual Laplacian. Clearly,

$$-U\Delta U^* = k^2.$$

Let $|\delta)(\delta|$ be the quadratic form given by

$$(f_1|\delta)(\delta|f_2) = \overline{f_1(0)}f_2(0).$$

Note that again it can be also written as

$$\int \overline{f(x)}\delta(x)g(x)\mathrm{d}x,$$

and thus is interpreted as a "potential". Let $(1|$ denote the functional on $L^2(\mathbb{R}^d)$ given by

$$(1|g) = \int g(k)\mathrm{d}k.$$

Using $\delta(x) = (2\pi)^{-d}\int \mathrm{e}^{\mathrm{i}kx}\mathrm{d}x$ we deduce that

$$U|\delta)(\delta|U^* = (2\pi)^{-d}|1)(1|.$$

Consider

$$U\left(-\Delta + \lambda|\delta)(\delta|\right)U^* = k^2 + \lambda(2\pi)^{-d}|1)(1|$$

as an example of an Aronszajn-Donoghue Hamiltonian. We compute:

$$\int \frac{\mathrm{d}^d k}{1+k^2} < \infty \;\Leftrightarrow\; d = 1,$$
$$\int \frac{\mathrm{d}^d k}{(1+k^2)^2} < \infty \;\Leftrightarrow\; d = 1, 2, 3,$$
$$\int \frac{\mathrm{d}^d k}{k^2(1+k^2)} < \infty \;\Leftrightarrow\; d = 3.$$

Thus

(1) for $d = 1$ it is of type IIb, so it can be defined in the form sense using the parameter λ (as we have already seen),

(2) for $d = 2$ it is of type IIIb. It can be renormalized.

(3) for d=3 it is of type IIIa. It can be renormalized.

(4) for $d \geq 4$ there is no nontrivial renormalization procedure.

Consider dimension $d = 2$. Let us compute the resolvent for $z = -p^2$. We have

$$\begin{aligned} g(-p^2) &= \gamma_{-1} + (p^2 - 1)(1|(H_0 + p^2)^{-1}(H_0 + 1)^{-1}|1) \\ &= \gamma_{-1} + (p^2 - 1)\int \frac{\mathrm{d}^3 k}{(k^2 + p^2)(k^2 + 1)} \;=\; \gamma_{-1} + \pi \ln p^2. \end{aligned}$$

Using that the Fourier transform of $k \mapsto \frac{1}{k^2+p^2}$ equals $x \mapsto 2\pi K_0(p|x|)$, where K_0 is the 0th MacDonald function, we obtain the following expression for the integral kernel of $(p^2 + H)^{-1}$:

$$2\pi K_0(p|x - y|) + \frac{K_0(p|x|)K_0(p|y|)}{\gamma_{-1} + \pi \ln p^2}. \tag{9.7.3}$$

In the physics literature one usually introduces the parameter $a = \mathrm{e}^{\gamma_{-1}/2\pi}$ called the scattering length. There is a bound state $K_0(|x|/a)$ with eigenvalue $-a^{-2}$.

Note that

$$\{f \in (1-\Delta)^{-1}L^2(\mathbb{R}^2) : \ f(0) = 0\} \tag{9.7.4}$$

is a closed subspace of $(1-\Delta)^{-1}L^2(\mathbb{R}^2)$. The domain of H is spanned by (9.7.4) and

$$(-a^{-2} - \Delta)^{-1}|1), \tag{9.7.5}$$

which is in $L^2(\mathbb{R}^2)\backslash(1-\Delta)^{-1}L^2(\mathbb{R}^2)$. In the position representation (9.7.5) is $x \mapsto 2\pi K_0(|x|/a)$ Around $r \sim 0$ we have the asymptotics $K_0(r) \simeq -\log(r/2) - \gamma$. Therefore, the domain of H contains functions that behave at zero as $C\big(\log(|x|/2a) + \gamma\big)$.

Consider dimension $d = 3$. Let us compute the resolvent for $z = -p^2$. We have

$$\begin{aligned} g(-p^2) &= \gamma_0 + p^2(1|(H_0+p^2)^{-1}H_0^{-1}|1) \\ &= \gamma_0 + p^2\int \frac{\mathrm{d}^3k}{(k^2+p^2)k^2} \ = \ \gamma_0 + p4\pi^2 \end{aligned}$$

Using that the Fourier transform of $k \mapsto \frac{1}{k^2+p^2}$ equals $x \mapsto 2\pi^2 \frac{\mathrm{e}^{p|x|}}{|x|}$, we obtain the following expression for the integral kernel of $(p^2 + H)^{-1}$:

$$2\pi^2 \frac{\mathrm{e}^{-p|x-y|}}{|x-y|} + \frac{\pi \mathrm{e}^{-p|x|}\mathrm{e}^{-p|y|}}{2(\gamma_0 + 4\pi^2 p)|x||y|}. \tag{9.7.6}$$

In the physics literature one usually introduces the parameter $a = -(4\pi\gamma_0)^{-1}$ called the scattering length.

$$\{f \in (1-\Delta)^{-1}L^2(\mathbb{R}^3) : \ f(0) = 0\} \tag{9.7.7}$$

is a closed subspace of $(1-\Delta)^{-1}L^2(\mathbb{R}^3)$. The domain of H is spanned by (9.7.7)

$$(a\mathrm{e}^{\mathrm{i}\pi/4} - \mathrm{i})(\mathrm{i} - \Delta)^{-1}|1) + (a\mathrm{e}^{-\mathrm{i}\pi/4} + \mathrm{i})(-\mathrm{i} - \Delta)^{-1}|1) \tag{9.7.8}$$

In the position representation $(\pm\mathrm{i} - \Delta)^{-1}|1)$ equals $x \mapsto 2\pi^2 \frac{\exp(\mathrm{e}^{\pm\mathrm{i}\pi/4}|x|)}{|x|}$. Therefore, the Hamiltonian with the scattering length a has the domain whose elements around zero behave as $C(1 - a/|x|)$.

For $a>0$ there is a bound state $\frac{\mathrm{e}^{-|x|/a}}{|x|}$ with eigenvalue $-a^{-2}$. To get the domain, instead of (9.7.8), we can adjoin this bound state to (9.7.7).

Note that the Hamiltonian is increasing wrt $\gamma_0 \in]-\infty,\infty]$. It is also increasing wrt a separately on $[-\infty,0]$ and $]0,\infty]$. At 0 the monotonicity is lost. $a=0$ corresponds to the usual Laplacian.

The following theorem summarizes a part of the above results.

Theorem 9.7.1 *Consider* $-\Delta$ *on* $C_{\mathrm{c}}^\infty(\mathbb{R}^d\backslash\{0\})$

(1) *It has the defficiency index* $(2,2)$ *for* $d=1$*.*

(2) *It has the defficiency index* $(1,1)$ *for* $d=2,3$*.*

(3) *It is essentially self-adjoint for* $d\geq 4$*.*

(4) *For* $d=1$ *its Friedrichs extension is* $-\Delta_{\mathrm{D}}$ *and its Krein extension is* $-\Delta$*.*

(5) *For* $d=2$ *its Friedrichs and Krein extension is* $-\Delta$*.*

(6) *For* $d=3$ *its Friedrichs extension is* $-\Delta$ *an its Krein extension corresponds to* $a=\infty$*.*

Let us sketch an alternative approach. The Laplacian in d dimensions written in spherical coordinates equals

$$\Delta = \partial_r^2 + \frac{d-1}{r}\partial_r + \frac{\Delta_{\mathrm{LB}}}{r^2},$$

where Δ_{LB} is the Laplace-Beltrami operator on the sphere. For $d \geq 2$, the eigenvalues of Δ_{LB} are $-l(l+d-2)$, for $l = 0, 1, \ldots$. For $d = 1$ instead of the Laplace-Beltrami operator we consider the parity operator with the eigenvalues ± 1. We will write $l = 0$ for parity $+1$ and $l = 1$ for parity -1. Hence the radial part of the operator is

$$\partial_r^2 + \frac{d-1}{r}\partial_r - \frac{l(l+d-2)}{r^2}.$$

The indicial equation of this operator reads

$$\lambda(\lambda + d - 2) - l(l + d - 2) = 0.$$

It has the solutions $\lambda = l$ and $\lambda = 2 - l - d$.

For $l \geq 2$ only the solutions behaving as r^l around zero are locally square integrable, the solutions behaving as r^{2-1-d} have to be discarded. For $l = 0, 1$ we have the following possible square integrable behaviors around zero:

	$l=0$	$l=1$	$l \geq 2$
$d=1$	r^0, r^1	r^0, r^1	$--$
$d=2$	$r^0, r^0 \ln r$	r^1	r^l
$d=3$	r^0, r^{-1}	r^1	r^l
$d \geq 4$	r^0	r^1	r^l

In dimension $d=1$ in both parity sectors we have non-uniqueness of boundary conditions. In dimensions $d=2,3$ this non-uniqueness appears only in the spherically symmetric sector. There is no nonuniqueness in higher dimensions.

9.8 Approximating delta potentials by separable potentials

Set $1_\Lambda(k) := 1\!\!1_{[0,\Lambda]}(|k|)$. The Laplacian with a delta potential can be conveniently approximated by a separable potential

$$-\Delta + \frac{\lambda}{(2\pi)^d}|1_\Lambda)(1_\Lambda|. \tag{9.8.9}$$

In dimension $d = 1$ and $d = 2$ (9.8.9) has a (single) negative bound state iff $\lambda < 0$.

Clearly, in dimension $d = 1$ (9.8.9) converges to $-\Delta + \lambda\delta$ in the norm resolvent sense for all $\lambda \in \mathbb{R}$.

It is easy to check that

$$-\Delta - \big(\gamma_{-1} + \pi \log(1 + \Lambda^2)\big)^{-1}|1_\Lambda)(1_\Lambda| \tag{9.8.10}$$

converges to $-\Delta_{(\gamma_{-1})}$ for all $\gamma_{-1} \in \mathbb{R}$.

In dimension $d = 3$ (9.8.9) has a (single) negative bound state for all $\frac{\lambda}{(2\pi)^3} < -(\Lambda 4\pi)^{-1}$. It is easy to check that

$$-\Delta - \big(\gamma_0 + 4\pi\Lambda\big)^{-1}|1_\Lambda)(1_\Lambda| \tag{9.8.11}$$

converges to $-\Delta_{(\gamma_0)}$ for all $\gamma_0 \in \mathbb{R}$.

Chapter 10

Orthogonal polynomials

First we discuss some basic general facts about orthogonal polynomials. Then we will classify the so called classical orthogonal polynomials, that is orthogonal polynomials that are eigefunctions of a certain second order differential operator. We will show that all classical orthogonal polynomials essentially fall into one of the following 3 classes:

(1) Hermite polynomials $H_n(x) = \frac{(-1)^n}{n!} \mathrm{e}^{x^2} \partial_x \mathrm{e}^{-x^2}$, which form an orthogonal basis in $L^2(\mathbb{R}, \mathrm{e}^{-x^2})$ and satisfy

$$(\partial_x^2 - 2x\partial_x + 2n)H_n(x) = 0.$$

(2) Laguerre polynomials $L_n^\alpha(x) = \frac{1}{n!} \mathrm{e}^x \partial_x^n \mathrm{e}^{-x} x^{n+\alpha}$, which form an orthogonal basis in $L^2(]0, \infty[, \mathrm{e}^{-x} x^\alpha)$ for $\alpha > -1$ and satisfy

$$(x\partial_x^2 + (\alpha + 1 - x)\partial_x + n)L_n^\alpha(x) = 0.$$

(3) Jacobi polynomials $P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\partial_x^n(1-x)^{\alpha+n}(1+x)^{\beta+n}$, which form an orthogonal basis in $L^2(]-1,1[, (1-x)^\alpha(1+x)^\beta)$ for $\alpha, \beta > -1$ and satisfy

$$(1-x^2)\partial_x^2 + (\beta - \alpha - (\alpha+\beta+2)x)\partial_x + n(n+\alpha+\beta+1)P_n^{\alpha,\beta}(x) = 0.$$

An important role in the proof is played by unbounded operators. More precisely, we use the fact that eigenvectors of hermitian operators with distinct eigenvalues are orthogonal.

Note that the proof is quite elementary – it has been routinely used in courses for physics students of 2nd year of University of Warsaw. In particular, one does not need to introduce the concept of a self-adjoint or essentially self-adjoint operator: one can limit oneself to the concept of a hermitian operator, which is much less technical and acceptable for students without sophisticated mathematical training.

10.1 Orthogonal polynomials

Let $-\infty \leq a < b \leq \infty$. Let $\rho > 0$ be a fixed positive integrable function on $]a, b[$ called a weight. Let x denote the generic variable in $\mathbb{R}$.

We will denote by Pol the space of complex polynomials of the real variable. We assume that

$$\int_a^b |x|^n \rho(x) \mathrm{d}x < \infty, \quad n = 0, 1, \dots . \tag{10.1.1}$$

Then Pol is contained in $L^2([a, b], \rho)$.

The monomials $1, x, x^2, \dots$ form a linearly independent sequence in $L^2([a, b], \rho)$. Applying the Gram-Schmidt orthogonalization to this sequence we obtain the orthogonal polynomials $P_0, P_1, P_2, \ldots$. Note that $\deg P_n = n$. There exist a simple criterion that allows us to check whether this is an orthogonal basis.

Theorem 10.1.1 *Suppose that there exists $\epsilon > 0$ such that*

$$\int_a^b \mathrm{e}^{\epsilon|x|}\rho(x)\mathrm{d}x < \infty.$$

Then Pol is dense in $L^2([a,b],\rho)$. Therefore, $P_0, P_1, \ldots$ form an orthogonal basis of $L^2([a,b],\rho)$.

Proof. Let $h \in L^2([a,b],\rho)$. Then for $|\mathrm{Im} z| \leq \frac{\epsilon}{2}$

$$\int_a^b |\rho(x)h(x)\mathrm{e}^{\mathrm{i}xz}|\mathrm{d}x \;\leq\; \left(\int_a^b \rho(x)\mathrm{e}^{\epsilon|x|}\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_a^b \rho(x)|h(x)|^2\mathrm{d}x\right)^{\frac{1}{2}} < \infty.$$

Hence, for $|\mathrm{Im} z| \leq \frac{\epsilon}{2}$ we can define

$$F(z) := \int_a^b \rho(x)\mathrm{e}^{-\mathrm{i}zx}h(x)\mathrm{d}x.$$

F is analytic in the strip $\{z \in \mathbb{C} \ : \ |\mathrm{Im} z| < \frac{\epsilon}{2}\}$. Let $(x^n|h) = 0$, $n = 0, 1, \dots$. Then

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} F(z)\Big|_{z=0} = (-\mathrm{i})^n \int_a^b x^n \rho(x) h(x) \mathrm{d}x = (-\mathrm{i})^n (x^n|h) = 0.$$

But an analytic function vanishing with all derivatves at one point vanishes in its whole (connected) domain. Hence $F = 0$ in the whole strip, and in particular on the real line. Hence $\hat{h} = 0$. Applying the inverse Fourier transformation we obtain $h = 0$.

Therefore, there are no nonzero vectors orthogonal to Pol. $\square$

10.2 Reminder about hermitian operators

In this chapter we will need some minimal knowledge about hermitian operators. In order to make it essentially self-contained, we recall that an operator A is hermitian if

$$(w|Av) = (Aw|v), \quad v, w \in \mathrm{Dom}\, A.$$

Theorem 10.2.1 *Let A be a hermitian operator.*

(1) *If* $v \in \mathrm{Dom}\, A$ *is its eigenvector with eigenvalue* λ*, that is* $Av = \lambda v$*, then* $\lambda \in \mathbb{R}$*.*

(2) *If* $\lambda_1 \neq \lambda_2$ *are its eigenvalues with eigenvectors* v_1 *and* v_2*, then* v_1 *is orthogonal to* v_2*.*

Proof. To prove (1), we note that

$$\lambda(v|v) = (v|Av) = (Av|v) = \overline{\lambda}(v|v).$$

then we divide by $(v|v) \neq 0$.

Proof of (2):

$$(\lambda_1 - \lambda_2)(v_1|v_2) = (Av_1|v_2) - (v_1|Av_2) = (v_1|Av_2) - (v_1|Av_2) = 0.$$

□

Remark 10.2.2 *In finite dimension we can always find an orthonormal basis consisting of eigenvectors of a hermitian operators. In infinite dimension this is not always the case. If it happens then the operator is essentially self-adjoint.*

10.3 2nd order differential operators

A general 2nd order differential operator without a 0th order term can be written as

$$\mathcal{C} := \sigma(x)\partial_x^2 + \tau(x)\partial_x, \tag{10.3.2}$$

for some functions $\sigma(x)$ and $\tau(x)$.

It is often convenient to rewrite $\mathcal{C}$ in a different form. Let $\rho(x)$ satisfy

$$\sigma(x)\rho'(x) = (\tau(x) - \sigma'(x))\rho(x). \tag{10.3.3}$$

We have then

$$\mathcal{C} = \rho(x)^{-1}\partial_x\rho(x)\sigma(x)\partial_x. \tag{10.3.4}$$

The form (10.3.4) of the operator $\mathcal{C}$ is convenient for the study of its hermiticity.

To simplify the exposition, in the remaining part of this subsection we will assume that $a = 0$ and $b = \infty$, which will illustrate the two possible types of endpoints. The generalization to arbitrary $a < b$ will be obvious.

Theorem 10.3.1 *Assume (10.1.1). Suppose also that*

(1) ρ *and* σ *are real differentiable functions on* $]0,\infty[$ *and* $\rho > 0$*;*

(2) *at the boundaries of the interval we have*

$$\begin{aligned} \sigma(0)\rho(0) &= 0, \\ \lim_{x\to\infty} \sigma(x)\rho(x)|x|^n &= 0, \ n = 0, 1, 2, \ldots. \end{aligned}$$

Then $\mathcal{C}$ *as an operator on* $L^2([0,\infty[,\rho)$ *with domain* Pol *is hermitian.*

Proof.

$$
\begin{aligned}
(g|\mathcal{C}f) &= \int_0^\infty \rho(x)\overline{g}(x)\rho(x)^{-1}\partial_x\sigma(x)\rho(x)\partial_x f(x)\mathrm{d}x \\
&= \lim_{R\to\infty}\int_0^R \overline{g(x)}\partial_x\sigma(x)\rho(x)\partial_x f(x)\mathrm{d}x \\
&= \lim_{R\to\infty}\overline{g(x)}\rho(x)\sigma(x)f'(x)\Big|_0^R - \lim_{R\to\infty}\int_0^R (\partial_x\overline{g(x)})\sigma(x)\rho(x)\partial_x f(x)\mathrm{d}x \\
&= -\lim_{R\to\infty}\overline{g'(x)}\rho(x)\sigma(x)f(x)\Big|_0^R + \lim_{R\to\infty}\int_0^R (\partial_x\rho(x)\sigma(x)\partial_x\overline{g(x)})f(x)\mathrm{d}x \\
&= \int_0^\infty \rho(x)\overline{(\rho(x)^{-1}\partial_x\sigma(x)\rho(x)\partial_x g(x))}f(x)\mathrm{d}x = (\mathcal{C}g|f).
\end{aligned}
$$

$\square$

Self-adjoint operators of the form (10.3.4) are often called Sturm-Liouville operators.

10.4 Hypergeometric type operators

We are looking for 2nd order differential operators whose eigenfunctions are polynomials. This restricts severely the form of such operators.

Theorem 10.4.1 *Let*

$$\mathcal{C} := \sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta(z) \tag{10.4.5}$$

Suppose there exist polynomials P_0, P_1, P_2 *of degree* $0, 1, 2$ *respectively, satisfying*

$$\mathcal{C}P_n = \lambda_n P_n.$$

Then

(1) $\sigma(z)$ *is a polynomial of degree* ≤ 2*,*

(2) $\tau(z)$ *is a polynomial of degree* ≤ 1*,*

(3) $\eta(z)$ *is a polynomial of degree* ≤ 0 *(in other words, it is a number).*

Proof. $\mathcal{C}P_0 = \eta(z)P_0$, hence $\deg \eta = 0$.

$\mathcal{C}P_1 = \tau(z)P_1' + \eta P_1$, hence $\deg \tau \leq 1$.

$\mathcal{C}P_2 = \sigma(z)P_2'' + \tau(z)P_2'(z) + \eta P_2$, hence $\deg \sigma \leq 2$. □

Clearly, the number η can be included in the eigenvalue. Therefore, it is enough to consider operators of the form

$$\mathcal{C} := \sigma(z)\partial_z^2 + \tau(z)\partial_z, \tag{10.4.6}$$

where $\deg \sigma \leq 2$ and $\deg \tau \leq 1$. We will show that for a large class of (10.4.6) there exists for every $n \in \mathbb{N}$ a polynomial P_n of degree n that is an eigenfunction of (10.4.6).

The eigenvalue equation of (10.4.6), that is equations of the form

$$\big(\sigma(z)\partial_z^2 + \tau(z)\partial_z + \lambda\big)f(z) = 0,$$

will be called hypergeometric type equations. Solutions of these equations will be called hypergeometric type functions. Polynomial solutions will be called hypergeometric type polynomials.

10.5 Generalized Rodrigues formula

Some of the properties of hypergeometric type polynomials can be introduced in a unified way. Let ρ satisfy

$$\sigma(z)\partial_z\rho(z) = \left(\tau(z) - \sigma'(z)\right)\rho(z). \tag{10.5.7}$$

Note that ρ can be expressed by elementary functions.

Let us fix σ. We will however make explicit the dependence on ρ. The operator $\mathcal{C}(\rho)$ can be written as

$$\begin{aligned} \mathcal{C}(\rho) &= \rho^{-1}(z)\partial_z\sigma(z)\rho(z)\partial_z & (10.5.8) \\ &= \partial_z\rho^{-1}(z)\sigma(z)\partial_z\rho(z) - \tau' + \sigma''. & (10.5.9) \end{aligned}$$

The following is a generalization of the Rodrigues formula, originally given in the case of

Legendre polynomials:

$$P_n(\rho; z) := \frac{1}{n!}\rho^{-1}(z)\partial_z^n \sigma^n(z)\rho(z) \tag{10.5.10}$$

$$= \frac{1}{2\pi \mathrm{i}}\rho^{-1}(z)\int_{[0^+]} \sigma^n(z+t)\rho(z+t)t^{-n-1}\mathrm{d}t. \tag{10.5.11}$$

Theorem 10.5.1 *P_n is a polynomial, typically of degree n, more precisely its degree is given as follows:*

(1) *If $\sigma'' = \tau' = 0$, then $\deg P_n = 0$.*

(2) *If $\sigma'' \neq 0$ and $-\frac{2\tau'}{\sigma''} + 1 = m$ is a positive integer, then*

$$\deg P_n = \begin{cases} n, & n = 0, 1, \ldots, m; \\ n - m - 1, & n = m+1, m+2, \ldots. \end{cases}$$

(3) *Otherwise, $\deg P_n = n$.*

We have

$$\left(\sigma(z)\partial_z^2 + \tau(z)\partial_z\right) P_n(\rho;z) = (n\tau' + n(n-1)\frac{\sigma''}{2})P_n(\rho;z), \tag{10.5.12}$$

$$(\sigma(z)\partial_z + \tau(z) - \sigma'(z))\, P_n(\rho;z) = (n+1)P_{n+1}(\rho\sigma^{-1};z), \tag{10.5.13}$$

$$\partial_z P_n(\rho;z) = \left(\tau' + (n-1)\frac{\sigma''}{2}\right) P_{n-1}(\rho\sigma;z), \tag{10.5.14}$$

$$\frac{\rho(z+t\sigma(z))}{\rho(z)} = \sum_{n=0}^{\infty} t^n P_n(\rho\sigma^n;z). \tag{10.5.15}$$

Proof. Introduce the following creation and annihilation operators:

$$\begin{aligned}
\mathcal{A}^+(\rho) &:= \sigma(z)\partial_z + \tau(z) = \rho^{-1}(z)\partial_z\rho(z)\sigma(z), \\
\mathcal{A}^- &:= \partial_z.
\end{aligned}$$

Note that

$$
\begin{aligned}
\mathcal{C}(\rho) &= \mathcal{A}^{+}(\rho)\mathcal{A}^{-} \\
&= \mathcal{A}^{-}\mathcal{A}^{+}(\rho\sigma^{-1}) - \tau' + \sigma''.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathcal{C}(\rho)\mathcal{A}^{+}(\rho) &= A^{+}(\rho)A^{-}A^{+}(\rho) \\
&= A^{+}(\rho)\big(\mathcal{C}(\rho\sigma) + \tau'\big).
\end{aligned}
$$

Therefore, if $\mathcal{C}(\rho\sigma^{n})F_0 = \lambda_0 F_0$, then

$$
\begin{aligned}
& \mathcal{C}(\rho)\ A^{+}(\rho)\cdots A^{+}(\rho\sigma^{n-1})F_0 \\
= & \left(\lambda_0 + n\tau' + n(n-1)\frac{\sigma''}{2}\right)A^{+}(\rho)\cdots A^{+}(\rho\sigma^{-1})F_0.
\end{aligned}
$$

Using

$$\begin{aligned}
A^+(\rho) &= \rho^{-1}(z)\partial_z\rho(z)\sigma(z), \\
A^+(\rho\sigma) &= \rho^{-1}(z)\sigma^{-1}(z)\partial_z\rho(z)\sigma^2(z), \\
\cdots &= \cdots \\
A^+(\rho\sigma^{n-1}) &= \rho^{-1}(z)\sigma^{-(n-1)}\partial_z\rho(z)\sigma^n(z),
\end{aligned}$$

we obtain

$$A^+(\rho)\cdots A^+(\rho\sigma^{n-1})F_0 = \rho(z)^{-1}\partial_z^n\rho(z)\sigma^n(z)F_0(z).$$

Take $F_0 = 1$, for which $\lambda_0 = 0$. We then obtain (10.5.12). $\square$

10.6 Classical orthogonal polynomials as eigenfunctions of a Sturm-Liouville operator

We are looking for $-\infty \leq a < b \leq \infty$ and weights $]a,b[\ni x \mapsto \rho(x)$ with the following properties: There exist polynomials $P_0, P_1, \dots$ satisfying $\deg P_n = n$ which form an orthogonal basis of $L^2(]a,b[,\rho)$ and are eigenfunctions of a certain 2nd order differential operator $\mathcal{C} := \sigma(x)\partial_x^2 + \tau(x)\partial_x$, that is, for some $\lambda_n \in \mathbb{R}$

$$\left(\sigma(x)\partial_x^2 + \tau(x)\partial_x + \lambda_n\right) P_n(x) = 0. \tag{10.6.16}$$

In particular, we want $\mathcal{C}$ to be hermitian on Pol.

We know that one has to satisfy the following conditions:

(1) For any $n \in \mathbb{N}$

$$\int_a^b \rho(x)|x|^n \mathrm{d}x < \infty, \tag{10.6.17}$$

which guarantees that $\mathrm{Pol} \subset L^2(]a,b[,\rho)$.

(2) σ has to be a polynomial of degree at most 2 and τ a polynomial of degree at most 1.

(See Thm 10.4.1).

(3) The weight ρ has to solve

$$\sigma(x)\rho'(x) = (\tau(x) - \sigma'(x))\rho(x), \tag{10.6.18}$$

to be positive, σ has to be real. (See Thm 10.3.1 (1)).

(4) We have to check the boundary conditions

(i) If an endpoint, say, a is a finite number, we check whether $\rho(a)\sigma(a) = 0$.

(ii) If an endpoint is infinite, say $a = -\infty$, then

$$\lim_{x \to -\infty} |x|^n \sigma(x)\rho(x) = 0, \quad n = 0, 1, 2, \ldots.$$

(see Thm 10.3.1 (2).)

We will find all weighted spaces $L^2(]a, b[, \rho)$ satisfying the conditions (1)-(4). It will turn out that in all cases the condition

$$\int_a^b \mathrm{e}^{\epsilon|x|}\rho(x)\mathrm{d}x < \infty \tag{10.6.19}$$

for some $\epsilon > 0$ will hold, which will guarantee that we obtain an orthogonal basis (see Thm 10.1.1).

We will simplify our answers to standard forms

(1) by changing the variable $x \mapsto \alpha x + \beta$ for $\alpha \neq 0$;

(2) by dividing (both the differential equation and the weight) by a constant.

As a result, we will obtain all classical orthogonal polynomials.

10.7 Classical orthogonal polynomials for $\deg \sigma = 0$

We can assume that $\sigma(x) = 1$.

If $\deg \tau = 0$, then

$$\mathcal{C} = \partial_y^2 + c\partial_y.$$

It is easy to discard this case.

Hence $\deg\tau = 1$. Thus

$$\mathcal{C} = \partial_y^2 + (ay+b)\partial_y.$$

Let us set $x = \sqrt{\frac{|a|}{2}}\left(y + \frac{b}{a}\right)$. We obtain

$$\mathcal{C} = \partial_x^2 + 2x\partial_x, \qquad a > 0; \tag{10.7.20}$$

$$\mathcal{C} = \partial_x^2 - 2x\partial_x, \qquad a < 0. \tag{10.7.21}$$

Thus $\rho(x) = \mathrm{e}^{\pm x^2}$.

$\sigma(x)\rho(x) = \mathrm{e}^{\pm x^2}$ is never zero, hence the only possible interval is $]-\infty, \infty[$.

If $a > 0$, we have $\rho(x) = \mathrm{e}^{x^2}$, which is impossible because of (4ii).

If $a < 0$, we have $\rho(x) = \mathrm{e}^{-x^2}$ and the interval $]-\infty, \infty[$ is admissible, and even satisfes (10.6.19). We obtain Hermite polynomials

10.8 Classical orthogonal polynomials for $\deg \sigma = 1$

We can assume that $\sigma(y) = y$.

If $\deg \tau = 0$, then

$$\mathcal{C} = y\partial_y^2 + c\partial_y$$

Such a $\mathcal{C}$ always decreases the degree of a polynomial. Therefore, if P is a polynomial and $\mathcal{C}P = \lambda P$, then $\lambda = 0$. Hence $P(x) = x^{-c}$. Therefore, we do not obtain polynomials of all degrees as eigenfunctions.

Thus $\deg \tau = 1$. Hence, for $b \neq 0$,

$$y\partial_y^2 + (a + by)\partial_y. \tag{10.8.22}$$

After rescaling, we obtain the operator:

$$\mathcal{C} = -x\partial_x^2 + (-\alpha - 1 + x)\partial_x.$$

We compute: $\rho = x^\alpha \mathrm{e}^{-x}$. $\rho(x)\sigma(x) = x^{\alpha+1}\mathrm{e}^{-x}$ is zero only for $x = 0$ i $\alpha > -1$. The interval $[-\infty, 0]$ is eliminated by (4ii). The interval $[0, \infty]$ is admissible for $\alpha > -1$, and even it satisfies 10.6.19. We obtain Laguerre polynomials.

10.9 Classical orthogonal polynomials for $\deg \sigma = 2$, σ has a double root

We can assume that $\sigma(x) = x^2$.

If $\tau(0) = 0$, then

$$\mathcal{C} = x^2 \partial_x^2 + cx\partial_x.$$

x^n are eigenfunctions of this operator, but the weight $\rho(x) = x^{c-2}$ is not good.

Let us assume now that $\tau(0) \neq 0$. After rescaling we can suppose that

$$\tau(x) = 1 + (\gamma + 2)x.$$

This gives $\rho(x) = \mathrm{e}^{-\frac{1}{x}} x^{\gamma}$. The only point where $\rho(x)\sigma(x) = \mathrm{e}^{-\frac{1}{x}} x^{\gamma+2}$ can be zero is $x = 0$. Hence the only possible intervals are $]-\infty, 0[$ and $]0, \infty[$. Both are eliminated by (4ii).

10.10 Classical orthogonal polynomials for $\deg \sigma = 2$, σ has two roots

If both roots are imaginary, it suffices to assume that $\sigma(x) = 1 + x^2$. We can suppose that $\tau(x) = a + (b+2)x$. Then $\rho(x) = \mathrm{e}^{a \arctan x}(1+x^2)^b$. $\sigma(x)\rho(x)$ is nowhere zero and therefore the only admissble interval is $[-\infty, \infty]$. This has to be rejected, because $\lim_{|x|\to\infty} \rho(x)|x|^n(1+x^2) = \infty$ for large enough n.

Thus we can assume that the roots are real. It suffices to assume that $\sigma(x) = 1 - x^2$. Let

$$\tau(x) = \beta - \alpha - (\alpha + \beta - 2)x,$$

which corresponds to the operator

$$(1-x^2)\partial_x^2 + (\beta - \alpha - (\alpha + \beta - 2)x\partial_x,$$

We obtain $\rho(x) = |1-x|^\beta |1+x|^\alpha$. (4ii) eliminates the intervals $]-\infty, -1[$ and $]1, \infty[$. There remains only the interval $[-1, 1]$, which satisfies (4i) for $\alpha, \beta > -1$. We obtain Jacobi polynomials.

Chapter 11

Homogeneous Schrödinger operators

This chapter is partly based on the joint work with V.Georgescu and L.Bruneau. Some of the results (in particular (11.3.6)) were obtained independently by Pankrashkin and Richard.

11.1 Modified Bessel equation

The modified Bessel equation has the form

$$(z^2\partial_z^2 + z\partial_z - z^2 - m^2)v(z) = 0.$$

It is equivalent to the homogeneous Schrödinger equation with energy -1

$$\begin{aligned} & z^{\frac{1}{2}-2}\left(z^2\partial_z^2 + z\partial_z - z^2 - m^2\right)z^{-\frac{1}{2}} \\ = \ & \partial_z^2 + \left(1/4 - m^2\right)\frac{1}{z^2} - 1. \end{aligned}$$

For $m \in \mathbb{C}\backslash\{\ldots,-3,-2,-1\}$ we define the modified Bessel function $I_m(z)$ as the unique solution of the modified Bessel equation satisfying

$$I_m(z) \sim \left(\frac{z}{2}\right)^m \frac{1}{\Gamma(m+1)}, \ z \sim 0.$$

For $m = \ldots,-3,-2,-1$, we extend this definition by continuity in m. It turns out that

$$I_m(z) = I_{-m}, \quad m \in \mathbb{Z}.$$

We define the Macdonald or Basset function as the unique solution of the modified Bessel equation satisfying, for $|\arg z| > \pi - \epsilon$,

$$\lim_{|z|\to\infty} \frac{K_m(z)}{\frac{\mathrm{e}^{-z}\sqrt{\pi}}{\sqrt{2z}}} = 1.$$

We have the relations

$$\begin{aligned} K_{-m}(z) = K_m(z) &= \frac{\pi}{2\sin \pi m}(I_{-m}(z) - I_m(z)), && (11.1.1)\\ I_m(z) &= \frac{1}{\mathrm{i}\pi}\big(K_m(\mathrm{e}^{-\mathrm{i}\pi} z) - \mathrm{e}^{\mathrm{i}\pi m} K_m(z)\big). && (11.1.2) \end{aligned}$$

As $x \to 0$, we have

$$K_m(x) \sim \begin{cases} \mathrm{Re}\left(\Gamma(m)\left(\frac{2}{x}\right)^m\right) & \text{if} \quad \mathrm{Re}\, m = 0,\ m \neq 0; \\ -\ln\left(\frac{x}{2}\right) - \gamma & \text{if} \quad m = 0, \\ \frac{\Gamma(m)}{2}\left(\frac{2}{x}\right)^m & \text{if} \quad \mathrm{Re}\, m > 0; \\ \frac{\Gamma(-m)}{2}\left(\frac{x}{2}\right)^m & \text{if} \quad \mathrm{Re}\, m < 0. \end{cases} \qquad (11.1.3)$$

From a single solution we can generate a whole ladder of solutions:

$$
\begin{aligned}
\left(\frac{1}{z}\partial_z\right)^n z^m I_m(z) &= z^{m-n} I_{m-n}(z), \\
\left(\frac{1}{z}\partial_z\right)^n z^{-m} I_m(z) &= z^{-m-n} I_{m+n}(z).
\end{aligned}
$$

Analoguous identities hold for $K_m(z)$.

For $m = \pm\frac{1}{2}$ (and hence for all $m \in \mathbb{Z}+\frac{1}{2}$) the modified Bessel and the MacDonald functions can be expressed in terms of elementary functions:

$$
\begin{aligned}
I_{\frac{1}{2}}(z) &= \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sinh z \\
I_{-\frac{1}{2}}(z) &= = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cosh z, \\
K_{-\frac{1}{2}}(z) = K_{\frac{1}{2}}(z) &= \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \mathrm{e}^{-z}.
\end{aligned}
$$

11.2 Standard Bessel equation

Replacing z with $\pm \mathrm{i}z$ in the modified Bessel equation leads to the standard Bessel equation:

$$(z^2\partial_z^2 + z\partial_z + z^2 - m^2)v(z) = 0.$$

It is equivalent to the homogeneous Schrödinger equation with energy 1

$$\begin{aligned} & z^{\frac{1}{2}-2}\left(z^2\partial_z^2 + z\partial_z + z^2 - m^2\right) z^{-\frac{1}{2}} \\ = \ & \partial_z^2 + \left(1/4 - m^2\right)\frac{1}{z^2} + 1. \end{aligned}$$

For $m \in \mathbb{C}\backslash\{\dots, -3, -2, -1\}$ we define the Bessel function $J_m(z)$ as the unique solution of the Bessel equation satisfying

$$J_m(z) \sim \left(\frac{z}{2}\right)^m \frac{1}{\Gamma(m+1)}, \ z \sim 0.$$

For $m = \dots, -3, -2, -1$, we extend this definition by continuity in m. It turns out that

$$J_m(z) = (-1)^m J_{-m}, \quad m \in \mathbb{Z}.$$

It is simply related to the modified Bessel function:

$$J_m(z) \;=\; \mathrm{e}^{\pm \mathrm{i}\pi \frac{m}{2}} I_m(\mp \mathrm{i}z).$$

There are two Hankel functions. They can be defined as the unique functions satisfying the following asymptotic formulas are true for $-\pi + \delta < \arg z < 2\pi - \delta$, $\delta > 0$:

$$\lim_{z\to\infty} \frac{H_m^+(z)}{\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}z} \mathrm{e}^{-\frac{\mathrm{i}m\pi}{2} - \frac{\mathrm{i}\pi}{4}}} = 1,$$

$$\lim_{z\to\infty} \frac{H_m^-(z)}{\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}z} \mathrm{e}^{\frac{\mathrm{i}m\pi}{2} + \frac{\mathrm{i}\pi}{4}}} = 1.$$

Both are analytic continuations of the MacDonald function – one to the lower and the other to the upper part of the complex plane:

$$H_m^\pm(z) \;=\; \frac{\pm 2}{\mathrm{i}\pi} \mathrm{e}^{\mp \mathrm{i}\pi \frac{m}{2}} K_m(\mp \mathrm{i}z),$$

$$K_m(z) \;=\; \pm \frac{\mathrm{i}\pi}{2} \mathrm{e}^{\pm \mathrm{i}m\pi} H^\pm(\pm \mathrm{i}z).$$

Note the identities

$$
\begin{aligned}
&H^{\pm}_{-m}(z) = \mathrm{e}^{\pm m\pi\mathrm{i}} H^{\pm}_{m}(z),\\
&J_m(z) = \tfrac{1}{2}\left(H^{+}_{m}(z) + H^{-}_{m}(z)\right),\\
&J_{-m}(z) = \tfrac{1}{2}\left(\mathrm{e}^{m\pi\mathrm{i}} H^{+}_{m}(z) + \mathrm{e}^{-m\pi\mathrm{i}} H^{-}_{m}(z)\right),\\
&H^{\pm}_{m}(z) = \pm\tfrac{\mathrm{i}\mathrm{e}^{\mp m\pi\mathrm{i}} J_m(z) - \mathrm{i}J_{-m}(z)}{\sin m\pi},
\end{aligned}
$$

From a single solution we can generate a whole ladder of solutions:

$$
\begin{aligned}
\left(\frac{1}{z}\partial_z\right)^n z^m J_m(z) &= z^{m-n} J_{m-n}(z),\\
\left(-\frac{1}{z}\partial_z\right)^n z^{-m} J_m(z) &= z^{-m-n} J_{m+n}(z).
\end{aligned}
$$

Analogous identities hold for $H^{\pm}_{m}(z)$.

For $m = \pm\frac{1}{2}$ (and hence for all $m \in \mathbb{Z}+\frac{1}{2}$) the modified Bessel and the MacDonald functions can be expressed in terms of elementary functions:

$$
\begin{aligned}
J_{\frac12}(z) &= = \left(\frac{2}{\pi z}\right)^{\frac12}\sin z,\\
J_{-\frac12}(z) &= \left(\frac{2}{\pi z}\right)^{\frac12}\cos z,\\
H^{\pm}_{\frac12}(z) &= \left(\frac{2}{\pi z}\right)^{\frac12}\mathrm{e}^{\pm\mathrm{i}(z-\frac{\pi}{2})},\\
H^{\pm}_{-\frac12}(z) &= \left(\frac{2}{\pi z}\right)^{\frac12}\mathrm{e}^{\pm\mathrm{i}z}
\end{aligned}
$$

11.3 Homogeneous Schrödinger operators

Let U_τ be the group of dilations on $L^2[0,\infty[$, that is $(U_\tau f)(x) = \mathrm{e}^{\tau/2}f(e^\tau x)$. We say that an operator A is homogeneous of degree ν if $U_\tau A U_\tau^{-1} = \mathrm{e}^{-\nu\tau}A$.

Let $D := \frac12(xp+px)$ denote the generator of dilations, so that $U_\tau = \mathrm{e}^{-\mathrm{i}\tau D}$.

For $c \in \mathbb{C}$ consider the differential expression

$$\tilde{L}_c := -\partial_x^2 + (-1/4 + c)x^{-2}. \tag{11.3.4}$$

Clearly (11.3.4) is homogeneous of degree -2.

Let $L_c^{\min}$ and $L_c^{\max}$ be the minimal and maximal operators associated to it in $L^2(0,\infty)$. That means, $L_c^{\min}$ is the closure of $\tilde{L}_c$ on $C_{\mathrm{c}}^\infty]0,\infty[$, and

$$\mathrm{Dom}(L_m^{\max}) = \{f \in \mathcal{D}'[0,\infty[:\ \tilde{L}_c f \in L^2[0,\infty[\}.$$

It is clear that $L_c^{\min}$ and $L_c^{\max}$ are closed operators homogeneous of degree -2, $L_c^{\min}$ is hermitian for real c and

$$(L_c^{\min})^* = L_{\bar{c}}^{\max}, \quad L_c^{\min} \subset L_c^{\max}.$$

We choose $\xi \in C^\infty(\mathbb{R}_+)$ such that $\xi = 1$ on $[0,1]$ and 0 on $[2,\infty[$. If m is a complex number we set

$$\zeta_m(x) = x^{1/2+m}\xi(x).$$

Proposition 11.3.1 ζ_m *is square integrable if and only if* $\mathrm{Re} m > -1$*, and then it belongs to*

$\mathrm{Dom}\, L_{m^2}^{\max}$. *For* $\mathrm{Re}m > 1$, ζ_m *belongs also to* $\mathrm{Dom}\, L_{m^2}^{\min}$, *otherwise it does not.*

For $\mathrm{Re}(m) > -1$, we define H_m to be the operator $L_{m^2}^{\max}$ restricted to $\mathrm{Dom}(L_{m^2}^{\min}) + \mathbb{C}\zeta_m$. We will see that the family of operators $m \mapsto H_m$ possesses very good properties. The main tool in its investigation is its resolvent, which can be computed explicitly.

Theorem 11.3.2 (1) *The operators H_m are homogeneous of degree -2*

(2) $H_m = L_{m^2}^{\min} = L_{m^2}^{\max}$ *iff* $\mathrm{Re}m > 1$.

(3) *For any* $\mathrm{Re}(m) > -1$ *we have* $\mathrm{sp}(H_m) = [0, \infty[$.

(4) *Let $\lambda \in \mathbb{C}\backslash[0, \infty[$. Set $R_m(\lambda; x, y)$ to be the integral kernel of $(\lambda - H_m)^{-1}$. Then for* $\mathrm{Re}k > 0$ *we have*

$$R_m(-k^2; x, y) = \begin{cases} \sqrt{xy} I_m(kx) K_m(ky) & \text{if} \quad x < y, \\ \sqrt{xy} I_m(ky) K_m(kx) & \text{if} \quad x > y. \end{cases},$$

where I_m is the modified Bessel function and K_m is the MacDonald function.

(5) *The resolvent $(\lambda - H_m)^{-1}$ is an analytic function of m for $\mathrm{Re}m > -1$. As $\mathrm{Re}m$ approaches -1, its norm blows up.*

(6) *We have* $H_m^* = H_{\overline{m}}$. *In particular,* H_m *is self-adjoint and positive iff* m *is real.*

In the following theorem we describe the self-adjoint extensions of $L_c^{\min}$ for various real values of c.

Theorem 11.3.3 (1) *If* $1 \leq c$, *then* $L_c^{\min} = L_c^{\max} = H_m$ *with* $m = \sqrt{c}$ *is self-adjoint. In particular,* H_m *is essentially self-adjoint on* $C_c^\infty]0, \infty[$.

(2) *If* $0 < c < 1$, *then a subspace of* $\operatorname{Dom} L_c^{\max}$ *complementary to* $\operatorname{Dom} L_c^{\min}$ *is spanned by* ζ_m *and* ζ_{-m} *with* $m = \sqrt{c}$. *Self-adjoint extensions are obtained by adjoining to* $\operatorname{Dom} L_c^{\min}$

$$\cos\alpha\zeta_m + \sin\alpha\zeta_{-m}.$$

Among them we have H_m, *which is the Friedrichs extension of* $L_c^{\min}$, *and* H_{-m}, *which is its Krein extension.*

(3) *A subspace of* $\operatorname{Dom} L_0^{\max}$ *complementary to* $\operatorname{Dom} L_0^{\min}$ *is spanned by* ζ_0 *and* $\log x\zeta_0$. *Self-adjoint extensions are obtained by adjoining to* $\operatorname{Dom} L_0^{\min}$

$$\cos\alpha\zeta_0 + \sin\alpha\log x\zeta_0.$$

Among them there is H_0, which is both a Friedrichs and Krein extension of $L_0^{\min}$.

(4) *If $c \leq 0$, then a subspace of $\mathrm{Dom}\, L_c^{\max}$ complementary to $\mathrm{Dom}\, L_c^{\min}$ is spanned by $\zeta_{\mathrm{i}k}$ and $\zeta_{-\mathrm{i}k}$ with $k = \sqrt{-c}$. Self-adjoint extensions are obtained by adjoining to $\mathrm{Dom}\, L_c^{\min}$*

$$\mathrm{e}^{\mathrm{i}\alpha}\zeta_{\mathrm{i}k} + \mathrm{e}^{-\mathrm{i}\alpha}\zeta_{-\mathrm{i}k}.$$

$H_{-\mathrm{i}k}$ and $H_{\mathrm{i}k}$ are non-self-adjoint extensions of $L_c^{\min}$.

Theorem 11.3.4 (1) $0 \leq m \leq m' \Rightarrow H_m \leq H_{m'}$.

(2) $0 \leq m < 1 \Rightarrow H_{-m} \leq H_m$.

(3) *If* $0 \leq \arg m \leq \pi/2$, *then* $\mathrm{Num}(H_m) = \{z \ : \ 0 \leq \arg z \leq 2\arg m\}$.

(4) *If* $-\pi/2 \leq \arg m \leq 0$, *then* $\mathrm{Num}(H_m) = \{z \ : \ 2\arg m \leq \arg z \leq 0\}$.

(5) *If* $\pi/2 < |\arg m| < \pi$, *then* $\mathrm{Num}(H_m) = \mathbb{C}$.

In the following theorem we show how to compute various quantities closely related to the operators H_m. We restrict ourselves to the case of real m.

Theorem 11.3.5 (1) *For $0 < a < b < \infty$, the integral kernel of $1\!\!1_{[a,b]}(H_m)$ is*

$$1\!\!1_{[a,b]}(H_m)(x,y) \;=\; \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{xy} J_m(kx) J_m(ky) k \mathrm{d}k,$$

where J_m is the Bessel function.

(2) *Let $\mathcal{F}_m$ be the operator on $L^2[0,\infty]$ given by*

$$\mathcal{F}_m : f(x) \mapsto \int_0^\infty J_m(kx)\sqrt{kx} f(x) \mathrm{d}x \tag{11.3.5}$$

Up to an inessential factor, $\mathcal{F}_m$ is the so-called Hankel transformation. $\mathcal{F}_m$ is a unitary involution on $L^2[0,\infty]$ diagonalizing H_m, more precisely

$$\mathcal{F}_m H_m \mathcal{F}_m^{-1} = x^2.$$

It satisfies $\mathcal{F}_m \mathrm{e}^{\mathrm{i}tD} = \mathrm{e}^{-\mathrm{i}tD} \mathcal{F}_m$ for all $t \in \mathbb{R}$.

(3) *If $m,k>-1$ are real then the wave operators associated to the pair H_m, H_k exist and*

$$\Omega_{m,k}^{\pm} := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH_m}\mathrm{e}^{-\mathrm{i}tH_k} = \mathrm{e}^{\pm\mathrm{i}(m-k)\pi/2}\mathcal{F}_m\mathcal{F}_k = \mathrm{e}^{\pm\mathrm{i}(m-k)\pi/2}\frac{\Xi_k(D)}{\Xi_m(D)}. \tag{11.3.6}$$

where

$$\Xi_m(t) = \mathrm{e}^{\mathrm{i}\ln(2)t}\frac{\Gamma(\frac{m+1+\mathrm{i}t}{2})}{\Gamma(\frac{m+1-\mathrm{i}t}{2})}.$$

(4) *The scattering operator for the pair (H_m, H_k) is a scalar operator $S_{m,k} = \mathrm{e}^{\mathrm{i}\pi(m-k)}\mathbb{1}$.*

11.4 Factorization

For each complex number α let $\widetilde{A}_\alpha$ be the differential expression

$$\widetilde{A}_\alpha := -\mathrm{i}\partial_x + \mathrm{i}\frac{\alpha}{x}$$

acting on distributions on $\mathbb{R}_+$. Its restriction to $C_c^\infty]0,\infty[$ is a closable operator in $L^2[0,\infty[$ whose closure will be denoted $A_\alpha^{\min}$. This is the minimal operator associated to $\widetilde{A}_\alpha$. The maximal operator $A_\alpha^{\max}$ associated to $\widetilde{A}_\alpha$ is defined as the restriction of $\widetilde{A}_\alpha$ to $\mathrm{Dom}(A_\alpha^{\max}) := \{f \in L^2[0,\infty[\ :\ \widetilde{A}_\alpha f \in L^2[0,\infty[\}$.

The following properties of the operators $A_\alpha^{\min}$ and $A_\alpha^{\max}$ are easy to check:

(i) $A_\alpha^{\min} \subset A_\alpha^{\max}$,

(ii) $(A_\alpha^{\min})^* = A_{-\overline{\alpha}}^{\max}$ and $(A_\alpha^{\max})^* = A_{-\overline{\alpha}}^{\min}$,

(iii) $A_\alpha^{\min}$ and $A_\alpha^{\max}$ are homogeneous of degree -1.

Proposition 11.4.1 (1) *We have* $A_\alpha^{\min} = A_\alpha^{\max}$ *if and only if* $|\mathrm{Re}\alpha| \geq 1/2$.

(2) *Let* $\mathrm{Re}\alpha > -1/2$. *Then*

(i) $\mathrm{rs}(A_\alpha^{\max}) = \mathbb{C}_-$.

(ii) The map $\alpha \mapsto A_\alpha^{\max}$ *is holomorphic in the region* $\mathrm{Re}\alpha > -1/2$.

(iii) If $\mathrm{Re}\alpha \geq 0$ *then* $\mathrm{i}A_\alpha^{\max}$ *is the generator of a* C^0*-semigroup of contractions*

(1) *Let* $\mathrm{Re}\alpha < 1/2$. *Then*

(i) $\mathrm{rs}(A_\alpha^{\min}) = \mathbb{C}_+$.

(ii) The map $\alpha \mapsto A_\alpha^{\min}$ *is holomorphic in the region* $\mathrm{Re}\alpha < 1/2$.

(iii) if $\mathrm{Re}\alpha \leq 0$ *the operator* $-\mathrm{i}A_\alpha^{\min}$ *is the generator of a* C^0*-semigroup of contractions*

$$m \geq 1: \qquad H_m = A^*_{1/2+m} A_{1/2+m} \;=\; A^*_{1/2-m} A_{1/2-m}, \qquad H_0^1 = \mathcal{Q}(H_m),$$

$$H_m = L^{\min}_{m^2} = L^{\max}_{m^2};$$

$$0 < m < 1: \qquad H_m = A^*_{1/2+m} A_{1/2+m} \;=\; \left(A^{\min}_{1/2-m}\right)^* A^{\min}_{1/2-m} \qquad H_0^1 = \mathcal{Q}(H_m),$$

H_m is the Friedrichs ext. of $L^{\min}_{m^2}$;

$$m = 0: \qquad H_0 = A^*_{1/2} A_{1/2},$$

$H_0^1 + \mathbb{C}\zeta_0$ dense in $\mathcal{Q}(H_0)$,

H_0 is the Friedrichs and Krein ext. of $L^{\min}_0$;

$$-1 < m < 0: \qquad H_m = \left(A^{\max}_{1/2+m}\right)^* A^{\max}_{1/2+m}, \qquad H_0^1 + \mathbb{C}\zeta_m = \mathcal{Q}(H_m),$$

H_m is the Krein ext. of $L^{\min}_{m^2}$.

In the region $-1 < m < 1$ (which is the most interesting one), it is quite remarkable that for strictly positive m one can factorize H_m in two different ways, whereas for $m \leq 0$ only one factorization appears.

As an example, let us consider the case of the Laplacian $-\partial_x^2$, i.e. $m^2 = 1/4$. The operators

$H_{1/2}$ and $H_{-1/2}$ coincide with the Dirichlet and Neumann Laplacian respectively. One usually factorizes them as $H_{1/2} = P_{\min}^* P_{\min}$ and $H_{-1/2} = P_{\max}^* P_{\max}$, where $P_{\min}$ and $P_{\max}$ denote the usual momentum operator on the half-line with domain $\mathcal{H}_0^1[0, \infty[$ and $H^1[0, \infty[$ respectively. The above analysis says that, whereas for the Neumann Laplacian this is the only factorization of the form $S^* S$ with S homogeneous, in the case of the Dirichlet Laplacian one can also factorize it in the rather unusual following way

$$H_{1/2} = \left(P_{\min} + \mathrm{i}x^{-1}\right)^* \left(P_{\min} + \mathrm{i}x^{-1}\right).$$

11.5 H_m as a holomorphic family of closed operators

The definition (or actually a number of equivalent definitions) of a holomorphic family of bounded operators is quite obvious and does not need to be recalled. In the case of unbounded operators the situation is more subtle.

Suppose that Θ is an open subset of $\mathbb{C}$, $\mathcal{H}$ is a Banach space, and $\Theta \ni z \mapsto H(z)$ is a function whose values are closed operators on $\mathcal{H}$. We say that this is a holomorphic family of closed operators if for each $z_0 \in \Theta$ there exists a neighborhood Θ_0 of z_0, a Banach space $\mathcal{K}$ and a holomorphic family of injective bounded operators $\Theta_0 \ni z \mapsto A(z) \in B(\mathcal{K}, \mathcal{H})$ such that $\mathrm{Ran}\, A(z) = \mathcal{D}(H(z))$ and

$$\Theta_0 \ni z \mapsto H(z)A(z) \in B(\mathcal{K}, \mathcal{H})$$

is a holomorphic family of bounded operators.

We have the following practical criterion:

Theorem 11.5.1 *Suppose that $\{H(z)\}_{z \in \Theta}$ is a function whose values are closed operators on $\mathcal{H}$. Suppose in addition that for any $z \in \Theta$ the resolvent set of $H(z)$ is nonempty. Then $z \mapsto H(z)$ is a holomorphic family of closed operators if and only if for any $z_0 \in \Theta$ there exists $\lambda \in \mathbb{C}$ and a neighborhood Θ_0 of z_0 such that $\lambda \in \mathrm{rs}(H(z))$ for $z \in \Theta_0$ and $z \mapsto (H(z) - \lambda)^{-1} \in B(\mathcal{H})$ is holomorphic on Θ_0.*

The above theorem indicates that it is more difficult to study holomorphic families of closed operators that for some values of the complex parameter have an empty resolvent set.

It is interesting to note that $\Xi_m(D)$ is a unitary operator for all real values of m and

$$\Xi_m^{-1}(D)x^{-2}\Xi_m(D) \tag{11.5.7}$$

is a function with values in self-adjoint operators for all real m. $\Xi_m(D)$ is bounded and invertible also for all m such that $\mathrm{Re}m \neq -1, -2, \dots$. Therefore, the formula (11.5.7) defines an operator for all $\{m \mid \mathrm{Re}m \neq -1, -2, \dots\} \cup \mathbb{R}$. Clearly, for $\mathrm{Re}m > -1$, this operator function coincides with the operator H_m studied in this paper. Its spectrum is always equal to $[0, \infty[$ and it is analytic in the interior of its domain.

One can then pose the following question: does this operator function exetnd to a holomorphic function of closed operators on the whole complex plane?

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