Trochę matematyki: Seria 5

Ogólnie mamy pełną separację zmiennych

$[redacted]\ U = R(r)\,\Theta(\theta)\,\Psi(\varphi)$ i harmoniki sferyczne, ale w przypadku symetrii obrotowej wokół jednej osi (oś z) zostaje

$$U = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\left(A_l r^l + B_l / r^{l+1}\right) P_l(\cos\theta), \quad \sum_{l=0}^{\infty} P_l(x)\, t^l = \frac{1}{\sqrt{t^2 - 2xt + 1}}$$

Przydatny wzór: wkład ładunku punktowego

$$\frac{1}{|r - r'|} = \sum_{l=0}^{\infty} \frac{[\min(r, r')]^l}{[\max(r, r')]^{l+1}} P_l(\hat{r} \cdot \hat{r}') \rightarrow U = \frac{1}{4\pi\varepsilon_0}\frac{q}{|r - r'|}$$

$P_0 = 1$
$P_1 = x$
$P_2 = 3x^2 - 1$

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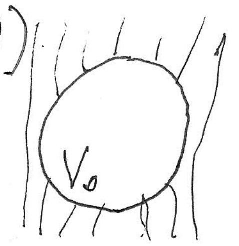
1) 

$$E_\infty = E_0 \hat{e}_z \Rightarrow U_\infty = E_0 z = E_0 r \cos\theta$$

$$U = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

Warunki brzegowe:

$$\mathrm{I} \rightarrow U(r=\infty) = E_0 r\cos\theta \Longrightarrow A_l = \begin{cases} 0 & l \neq 1 \\ 4\pi\varepsilon_0 E_0 & l = 1 \end{cases}$$

$$\mathrm{II} \rightarrow U(r=R) = V_0$$

$$\mathrm{III} \rightarrow \partial_\theta U(r=R) = 0 \Longrightarrow B_l(l \geq 2) = 0$$

$$\frac{1}{4\pi\varepsilon_0}\left(A_1 R\cos\theta + \frac{B_0}{R} + \frac{B_1}{R^2}\cos\theta\right) = V_0 \rightarrow \begin{cases} A_1 R = -B_1/R^2 \rightarrow B_1 = -A_1 R^3 \\ \frac{B_0}{R} = 4\pi\varepsilon_0 V_0 \end{cases}$$

$$\text{razem: } U = E_0 r\cos\theta + \frac{V_0 R}{r} - \frac{E_0 R^3}{r^2}\cos\theta = \underbrace{\frac{V_0 R}{r}}_{\text{naładow. kula}} + \overbrace{E_0 r\cos\theta}^{\text{zewn. pole}} - \underbrace{\frac{\vec{p}\cdot\vec{r}}{r^3}}_{\text{indukowany moment dipolowy}}$$

$\vec{p} = R^3 \vec{E}_0$

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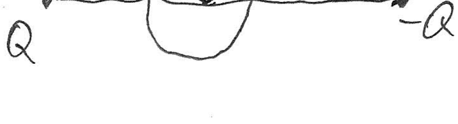
sprytniej: metoda obrazów

, czyli zadziała mocny i daleki dipol.



z $Q \cdot d^2 = \mathrm{const}$ i $d \to \infty$, wtedy w środku obrazy zlewają się w dipol

$$\left(E_0 = \frac{Q}{2\pi\varepsilon_0 d^2}\right)$$

$$q' = -\frac{RQ}{d},\ d' = \frac{R^2}{d} \rightarrow \vec{p}' = q'd' = \frac{R^3 Q}{d^2} = 2\pi\varepsilon_0 R^3 E_0$$

2) ↑ ↑$\vec{E}_y$ $\quad y \uparrow, \rightarrow x \quad$ $E_\infty = E_0 \hat{e}_y \rightarrow U_\infty = E_0 y = E_0 r \sin\varphi$

$$U = \ln\frac{r}{A_0} + B_0 + \sum_{m=1}^{\infty} \left(A_m r^m + B_m r^{-m}\right)\left(C_m \sin(m\varphi) + D_m \cos(m\varphi)\right)$$

warunki brzegowe:

I → $U(r=\infty) = U_\infty = E_0 \, r \sin\varphi \rightarrow \cancel{B_m} A_m(m>1) = 0,\ A_1 C_1 = E_0,\ D_1 = 0$

$B_0 = 0,\ \ln\frac{r}{A_0} = 0$

II → $U(\varphi = 0 \vee \varphi = \pi) = 0 \rightarrow D_m = 0$

III → $U(r=a) = 0 \rightarrow C_m(m>1) = 0,\ A_1 a + \frac{B_1}{a} = 0 \rightarrow B_1 C_1 = -E_0 a^2$

składamy razem: $\boxed{U = \left(E_0 r - \frac{E_0 a^2}{r}\right)\sin\varphi}$

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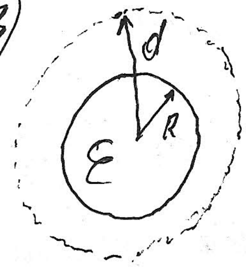
3)



mamy 3 obszary: $r \le R,\ R < r < d,\ r > d$

potencjał ładunku → $U_{ch} = \begin{cases} \frac{q}{4\pi\varepsilon_0 d} \sum_{m=0}^{\infty} \frac{r^m}{d^m} P_m(\cos\theta) & r<d \\ \frac{q}{4\pi\varepsilon_0 d} \sum_{m=0}^{\infty} \frac{d^m}{r^m} P_m(\cos\theta) & r>d \end{cases}$

potencjał kuli: $U_k = \begin{cases} \sum_{m=0}^{\infty} A_m r^m P_m(\cos\theta) & r<R \text{ — bez } B_m \text{ bo } U(r=0)=0 \\ \sum_{m=0}^{\infty} B_m r^{-(m+1)} P_m(\cos\theta) & r>R \text{ — bez } A_m \text{ bo } U(r=\infty)=0 \end{cases}$

zbieramy razem:
$$U = \begin{cases} \sum_{m=0}^{\infty} A_m r^m P_m(\cos\theta) & r<R \quad [U_I] \\ \sum_{m=0}^{\infty} P_m(\cos\theta)\left[\frac{B_m}{r^{m+1}} + \frac{q}{4\pi\varepsilon_0 d}\left(\frac{r}{d}\right)^m\right] & R<r<d \quad [U_{II}] \\ \sum_{m=0}^{\infty} P_m(\cos\theta)\left[\frac{B_m}{r^{m+1}} + \frac{q}{4\pi\varepsilon_0 d}\left(\frac{d}{r}\right)^m\right] & r>d \quad [U_{III}] \end{cases}$$

warunki brzegowe:

1) → $U_I(r=R) = U_{II}(r=R) \rightarrow \sum A_m R^m P_m(\cos\theta) = \sum\left[\frac{B_m}{R^{m+1}} + \frac{q}{4\pi\varepsilon_0 d}\left(\frac{R}{d}\right)^m\right] P_m(\cos\theta)$

2) $\varepsilon\, \partial_r U_I(r=R) = \partial_r U_{II}(r=R) \rightarrow \sum \varepsilon m A_m R^{m-1} P_m(\cos\theta) = \sum\left[\frac{B_m \cdot (-m-1)}{R^{m+2}} + \frac{q m}{4\pi\varepsilon_0 d^2}\left(\frac{R}{d}\right)^{m-1}\right] \cdot P_m(\cos\theta)$

powinny być spełnione dla każdego θ, czyli dla różnych m oddzielnie dla każdego m $\begin{cases} A_m R^m = \frac{B_m}{R^{m+1}} + \frac{q}{4\pi\varepsilon_0}\frac{R^m}{d^{m+1}} \\ \varepsilon m A_m R^{m-1} = -(m+1)\frac{B_m}{R^{m+2}} + \frac{q}{4\pi\varepsilon_0}\frac{m R^{m-1}}{d^{m+1}} \end{cases}$

$m=0 \rightarrow \begin{cases} A_0 = \frac{B_0}{R} + \frac{q}{4\pi\varepsilon_0 d} \\ 0 = -\frac{B_0}{R^2} \end{cases} \rightarrow B_0 = 0,\ A_0 = \frac{q}{4\pi\varepsilon_0 d}$

$m>0 \rightarrow \begin{cases} A_m R^{2m+1} = B_m + \frac{q}{4\pi\varepsilon_0}\frac{R^{2m+1}}{d^{m+1}} \\ \varepsilon m A_m R^{2m+1} = -B_m(m+1) + \frac{q}{4\pi\varepsilon_0}\frac{m R^{2m+1}}{d^{m+1}} \end{cases}$

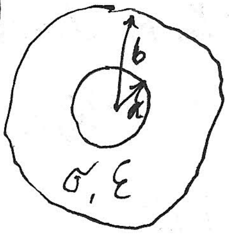
$$\varepsilon m B_m + \frac{q\cdot\varepsilon m}{4\pi\varepsilon_0}\frac{R^{2m+1}}{d^{m+1}} = -B_m(m+1) + \frac{qm}{4\pi\varepsilon_0}\frac{R^{2m+1}}{d^{m+1}}$$

$$B_m = -\frac{q}{4\pi\varepsilon_0}\frac{R^{2m+1}}{d^{m+1}}\frac{m(\varepsilon-1)}{m(\varepsilon+1)+1};\quad A_m = \frac{q}{4\pi\varepsilon_0}\frac{1}{d^{m+1}}\frac{2m+1}{m(\varepsilon+1)+1}$$

interesuje nas moment dipolowy $\rightarrow U \approx \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} = B_1\frac{\cos\theta\cdot r}{r^3}$

$\vec{p} = 4\pi\varepsilon_0 B_1 \hat{e}_z = \boxed{-q\frac{\varepsilon-1}{\varepsilon+2}\left(\frac{R}{d}\right)^3\cdot d\,\hat{e}_z}$ P.S. jak widać, metoda obrazów nie działa :(

4)



z Gaussa $E = \frac{Q}{4\pi\varepsilon\varepsilon_0 r^2} \rightarrow U = \frac{Q}{4\pi\varepsilon\varepsilon_0 r} \rightarrow \Delta U = \frac{Q}{4\pi\varepsilon\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$

$\vec{j} = \sigma\vec{E} \rightarrow I = \oiint \vec{j}\,d\vec{s} = \sigma\oiint \vec{E}\,d\vec{s} = \frac{\sigma Q}{\varepsilon\varepsilon_0}$ (przyda się i potem)

$R = \frac{\Delta U}{I} = \frac{1}{4\pi\sigma}\left(\frac{1}{a}-\frac{1}{b}\right)$

5) $I = \frac{\sigma Q}{\varepsilon\varepsilon_0} \rightarrow R\cdot C = \frac{U}{I}\cdot\frac{Q}{U} = \frac{\varepsilon\varepsilon_0}{\sigma}$, c.k.d.

6) Q σ $-Q$, V_0

$C = \frac{Q}{V_0}$; z poprzedniego $R = \frac{\varepsilon\varepsilon_0}{\sigma C} = \frac{\varepsilon\varepsilon_0 V_0}{\sigma Q}$

$\boxed{I = \frac{V_0}{R} = \frac{\sigma Q}{\varepsilon\varepsilon_0}}$, $W_J = \frac{V_0^2}{R} = \frac{\sigma}{\varepsilon\varepsilon_0}QV_0$, czyli energia Joule'a to energia z kondensatora

$I = \frac{dQ}{dt} = \frac{\sigma}{\varepsilon\varepsilon_0}Q \rightarrow \cancel{Q = Q_0/1-\varepsilon\varepsilon_0}$

$$\frac{dQ}{Q} = \frac{\sigma}{\varepsilon\varepsilon_0}dt \rightarrow \ln Q/Q_0 = \frac{\sigma}{\varepsilon\varepsilon_0}(t-t_0) \rightarrow Q = Q_0 e^{\left(1-\frac{\sigma}{\varepsilon\varepsilon_0}t\right)}$$