1) pojedyńczy walec:

z symetrii

$$\vec{E} = E\,\hat{e}_\rho$$

$$S = 2\,S_{podst} + S_{bok}$$

$m = \min(A, R)$
różna całka po ładunku wewnątrz i na zewnątrz

$$\vec{n}_{podst} = \pm\hat{e}_z \Rightarrow \iint_{podst} \vec{E}\,d\vec{S} = 0$$

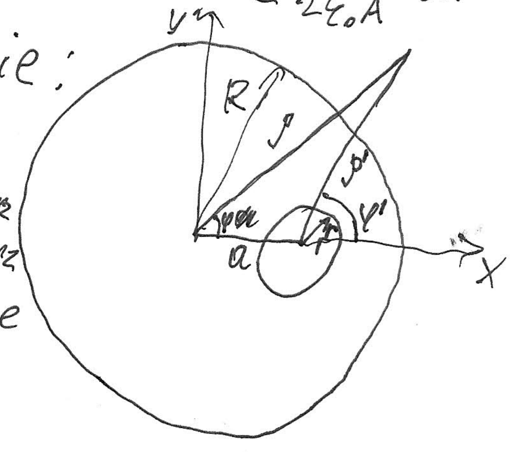
$$\oiint \vec{E}\,d\vec{S} = \iint_{bok} \vec{E}\,d\vec{S} = \int_0^{2\pi} d\varphi \int_0^L dz \Big|_{\rho=A} \rho\cdot E = 2\pi A L E$$

$$\iiint \frac{G}{\varepsilon_0}\,dV = \int_0^{2\pi} d\varphi \int_0^L dz \int_0^m d\rho\, \rho\cdot\frac{G}{\varepsilon_0} = \pi m^2 L \frac{G}{\varepsilon_0}$$

$$E = \frac{\pi m^2 \cancel{L} G}{2\pi A \cancel{L} \varepsilon_0} = \begin{cases} \dfrac{GA}{2\varepsilon_0} & \text{dla } A \le R \\ \dfrac{GR^2}{2\varepsilon_0 A} & \text{dla } A \ge R \end{cases} \rightarrow \vec{E} = \begin{cases} \dfrac{G\rho}{2\varepsilon_0}\,\hat{e}_\rho & \rho \le R \\ \dfrac{GR^2}{2\rho\varepsilon_0}\,\hat{e}_\rho & \rho \ge R \end{cases}$$

Zadanie:

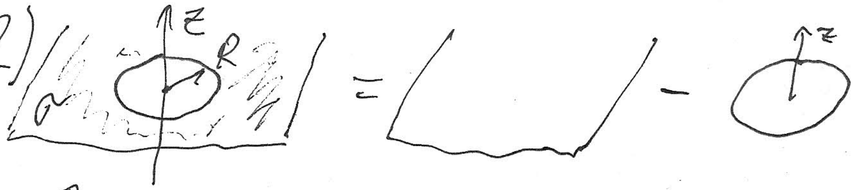
3 obszary
1) zewnątrz
2) wewnątrz
3) w dziurce

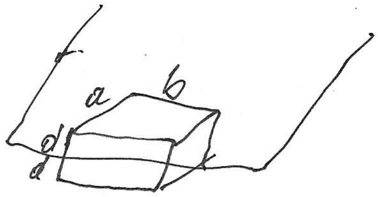


$$\vec{E} = \vec{E}_{duży} - \vec{E}_{mały} = \begin{cases} \dfrac{GR^2\hat{e}_\rho}{2\rho\varepsilon_0} - \dfrac{Gr^2\hat{e}_{\rho'}}{2\rho'\varepsilon_0} & \text{zewn.} \\ \dfrac{G\rho\hat{e}_\rho}{2\varepsilon_0} - \dfrac{Gr^2\hat{e}_{\rho'}}{2\rho'\varepsilon_0} & \text{wewn.} \\ \dfrac{G\rho\hat{e}_\rho}{2\varepsilon_0} - \dfrac{G\rho'\hat{e}_{\rho'}}{2\varepsilon_0} & \text{dziura} \end{cases}$$

$x = \rho\cos\varphi = \rho'\cos\varphi' + a$

$y = \rho\sin\varphi = \rho'\sin\varphi'$

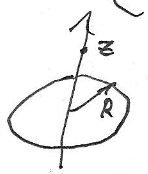
2) 

płaszczyzna: z tw. Gaussa 

z symetrii $\vec{E} = E\hat{e}_z$, więc przez powierzchnie boczne $\iint \vec{E}d\vec{s} = 0$

$$\oiint \vec{E}d\vec{s} = \iint_{\text{góra}} \vec{E}\,d\vec{s} + \iint_{\text{dół}} \vec{E}d\vec{s} = 2E\cdot ab \,;\ \iiint \frac{\rho}{\varepsilon_0}dV = \frac{\sigma}{\varepsilon_0}\cdot ab$$

$$E = \frac{\sigma}{2\varepsilon_0} \Rightarrow \vec{E}_{pł} = \begin{cases} \frac{\sigma}{2\varepsilon_0}\hat{e}_z & z>0 \\ \frac{-\sigma}{2\varepsilon_0}\hat{e}_z & z<0 \end{cases} \Longrightarrow U_{pł} = -\int E_z dz = \begin{cases} -\frac{\sigma z}{2\varepsilon_0} & z>0 \\ \frac{\sigma z}{2\varepsilon_0} & z<0 \end{cases}$$

dysk: z całki kulombowskiej 

$$\vec{r} = z\hat{e}_z \quad , \vec{r}' = \rho(\cos\varphi\, \hat{e}_x + \sin\varphi\, \hat{e}_y)$$

$z' = 0$, $\rho \in [0,R]$, $\varphi \in [0, 2\pi]$

$$|r - r'| = \sqrt{z^2+\rho^2}$$

$$U = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\varphi \int_0^R d\rho \frac{\rho\sigma}{\sqrt{\rho^2+z^2}} = \left| \begin{array}{l} w = \sqrt{\rho^2+z^2} \\ dw = \frac{\rho}{\sqrt{\rho^2+z^2}}d\rho \end{array} \right| = \frac{\sigma}{2\varepsilon_0}\left[w\right]_z^{\sqrt{R^2+z^2}} =$$

$$= \pm\frac{\sigma z}{2\varepsilon_0}\left(\sqrt{1+\frac{R^2}{z^2}} - 1\right)$$ (znak zależy od znaku z)

dla $z>0$ $\quad U_\odot = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2} - z\right) \rightarrow \vec{E}_\odot = -\nabla U_\odot = \frac{-\sigma}{2\varepsilon_0}\hat{e}_z\left(\frac{z}{\sqrt{z^2+R^2}} - 1\right)$

$$U = U_{pł} - U_\odot = \frac{-\sigma\sqrt{z^2+R^2}}{2\varepsilon_0} \,,\ \vec{E} = \vec{E}_{pł} - \vec{E}_\odot = \frac{\sigma z\,\hat{e}_z}{2\varepsilon_0\sqrt{z^2+R^2}}$$

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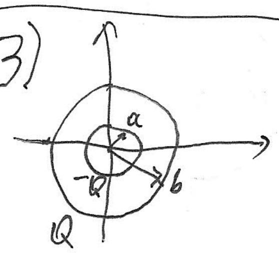
3) 

$\lambda_1 = \frac{-Q}{2\pi a}$ $\quad$ $\lambda_2 = \frac{Q}{2\pi b}$

monopol: $\iiint \rho\, dV' = \int_0^{2\pi} d\varphi \Big|_{\rho=a} \rho\lambda_1 + \int_0^{2\pi} d\varphi \Big|_{\rho=b} \rho\lambda_2 =$

$= -\frac{2\pi Q}{2\pi a}\cdot a + \frac{2\pi Q}{2\pi b}\cdot b = 0$ ~ nie ma monopola

dipol: $p_i = \iiint \rho\, r_i\, dV \longrightarrow z' = 0 \rightarrow p_z = \iiint \rho\cdot z\, dV = 0$

$$p_x = \int_0^{2\pi} d\varphi \Big|_{\rho=a} \rho\cdot\lambda_1 x + \int_0^{2\pi} d\varphi \Big|_{\rho=b} \rho\cdot\lambda_2 x = \int_0^{2\pi} d\varphi \left[\frac{-Q}{2\pi}\cdot a\cos\varphi + \frac{Q}{2\pi}\cdot b\cos\varphi\right] = 0$$

$$p_y = \int_0^{2\pi} d\varphi \left[-\frac{Qa}{2\pi} + \frac{Qb}{2\pi}\right]\sin\varphi = 0$$ — nie ma dipola

$$Q_{ij} = \iiint \rho \left(3 r_i r_j - |r|^2 \delta_{ij}\right) dV$$

$$Q_{xx} = \int_0^{2\pi} d\varphi \, \rho \lambda_1 (3x^2 - r^2)\Big|_{\substack{\rho=a \\ z=0}} + \int_0^{2\pi} d\varphi \, \rho \lambda_2 (3x^2 - r^2)\Big|_{\substack{\rho=b \\ z=0}} =$$

$$= \int_0^{2\pi} d\varphi \left[\frac{-Q}{2\pi a} \cdot a^3 (3\cos^2\varphi - 1) + \frac{Q}{2\pi b} \cdot b^3 (3\cos^2\varphi - 1) \right] =$$

$$= \frac{Q(b^2-a^2)}{2\pi} \int_0^{2\pi} d\varphi \, (3\cos^2\varphi - 1) = \left| \begin{array}{l} \cos^2\varphi = \frac{1+\cos 2\varphi}{2} \\ \sin^2\varphi = \frac{1-\cos 2\varphi}{2} \end{array} \right| =$$

$$= \frac{Q(b^2-a^2)}{2\pi} \int_0^{2\pi} d\varphi \left(\frac{1}{2} + \frac{3}{2}\cos 2\varphi \right) = \boxed{\frac{Q(b^2-a^2)}{2}}$$

analogicznie $Q_{yy} = \boxed{\frac{Q(b^2-a^2)}{2}}$

$$Q_{zz} = \int_0^{2\pi} d\varphi \left[-\frac{Q}{2\pi a} \cdot a + \frac{Q}{2\pi b} \cdot b \right] (3z^2 - r^2) = \int_0^{2\pi} d\varphi \left[-\frac{Q}{2\pi} \cdot (-\rho^2)_{\rho=a} + \frac{Q}{2\pi}(-\rho^2)_{\rho=b} \right]$$

$= \boxed{Q(a^2-b^2)} \longrightarrow$ sprawdzamy: $Q_{xx} + Q_{yy} + Q_{zz} = 0$

$$Q_{xy} = \int_0^{2\pi} d\varphi \, \rho \lambda_1 \cdot 3xy \Big|_{\substack{\rho=a \\ z=0}} + \int_0^{2\pi} d\varphi \, \rho \lambda_2 \cdot 3xy \Big|_{\substack{\rho=b \\ z=0}} = \int_0^{2\pi} d\varphi \left[\frac{-Q}{2\pi a} \rho^3 \right]_{\rho=a} (3\sin\varphi\cos\varphi)$$

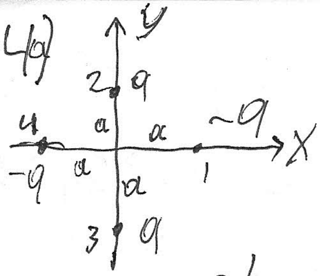
$$+ \int_0^{2\pi} d\varphi \left[\frac{Q}{2\pi b} \rho^3 \right]_{\rho=b} (3\sin\varphi\cos\varphi) = \frac{3Q}{2\pi}(b^2-a^2) \int_0^{2\pi} d\varphi \, \frac{1}{2}\sin 2\varphi = 0$$

analogicznie $Q_{xz} = Q_{yz} = 0 \longrightarrow \overline{\overline{Q}} = Q(b^2-a^2) \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$U = \frac{1}{4\pi\varepsilon_0 r^5} \vec{r} \cdot \overline{\overline{Q}} \cdot \vec{r} = \frac{Q(b^2-a^2)}{4\pi\varepsilon_0 r^5} (x \ y \ z) \begin{pmatrix} \frac{1}{2} & & 0 \\ & \frac{1}{2} & \\ 0 & & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$= \frac{Q(b^2-a^2)}{8\pi\varepsilon_0 r^5} (x^2 + y^2 - 2z^2) = \boxed{\frac{Q(b^2-a^2)}{8\pi\varepsilon_0 r^5} \left(r^2 - 3z^2\right)}$$

4a)

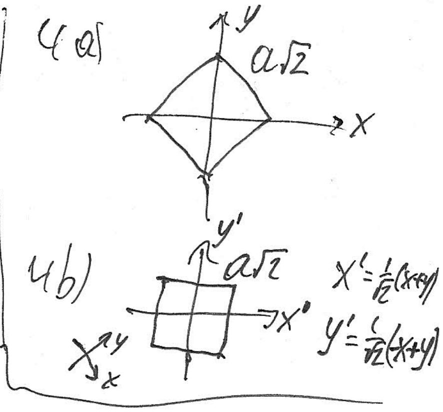


$$Q_{ij} = \sum q_i (3 r_i r_j - \delta_{ij} |r|^2)$$

$$Q_{xx} = \sum q_i (3x^2 - r^2) =$$

$z = 0!$

$$= q\left[-(3a^2 - a^2) + (3\cdot 0 - a^2) + (3\cdot 0 - a^2) - (3(-a)^2 - a^2)\right]$$



$$Q_{xx} = -6qa^2$$

$$Q_{xy} = \sum q_i \cdot 3xy = 3q\left[-a\cdot 0 + 0\cdot a + 0\cdot(-a) - (-a)\cdot 0\right] = 0$$

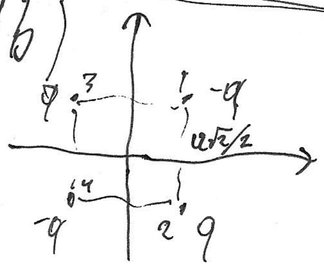
$$Q_{yy} = \sum q_i (3y^2 - r^2) = q\left[-(3\cdot 0^2 - a^2) + (3a^2 - a^2) + (3a^2 - a^2) - (-a^2)\right] = 6qa^2$$

$$Q_{xz} = \sum q_i \cdot 3xz = Q_{yz} = 0$$

$$Q_{zz} = \sum q_i (3z^2 - r^2) = q\left[-(-a^2) + (-a^2) + (-a^2) - (-a^2)\right] = 0$$

$$\bar{\bar{Q}} = 6qa^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad U = \frac{1}{4\pi\varepsilon_0} \vec{r}\,\bar{\bar{Q}}\,\vec{r} = \frac{3qa^2}{2\pi\varepsilon_0}(y^2 - x^2)$$

4b)



$$Q_{xx} = q\left[-\left(3\frac{a^2}{2} - a^2\right) + \left(3\frac{a^2}{2} - a^2\right) + \left(3\left(\frac{-a}{\sqrt{2}}\right)^2 - a^2\right) - \left(3\left(\frac{-a}{\sqrt{2}}\right)^2 - a^2\right)\right] = 0$$

$Q_{yy} = Q_{zz} = 0$ analogicznie

$Q_{xz} = Q_{yz} = 0$, bo $z = 0$

$$Q_{xy} = Q_{yx} = 3q\left[-\frac{a^2}{2} + \left(-\frac{a^2}{2}\right) + \left(-\frac{a^2}{2}\right) - \frac{a^2}{2}\right] = -6qa^2$$

$$\bar{\bar{Q}} = 6qa^2 \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad U = \frac{1}{4\pi\varepsilon_0} \cdot 6qa^2 \cdot (-2x'y') = \left| \begin{matrix} x' = \frac{1}{\sqrt{2}}(x+y) \\ y' = \frac{1}{\sqrt{2}}(y-x) \end{matrix} \right| = \frac{3qa^2}{2\pi\varepsilon_0}(y^2 - x^2)$$

zgodne z 4a

:)

$$\int_{-1}^{1} (1 - 3w^2)\,dw = \left[w - w^3\right]_{-1}^{1} = 0$$

za chwilę się przyda

5) $U = \frac{1}{4\pi\varepsilon_0 r^5} Q_{ij} r_i r_j$; $\vec{E} = -\vec{\nabla} U$;

S.3

strumień przez powierzchnię sfery → $\vec{n} = \hat{e}_r$, $\vec{E}\cdot\hat{e}_r = E_r$

$$E_r = \frac{-\hat{e}_r\cdot\hat{e}_r}{h_r}\partial_r U = -\partial_r U = \frac{-Q_{ij}}{4\pi\varepsilon_0}\partial_r \frac{r_i r_j}{r^5} = \left| \partial_r r_i = \frac{r_i}{r} \right| =$$

$$= \frac{-Q_{ij}}{4\pi\varepsilon_0}\,\frac{2 r_i r_j \cdot \frac{r^5}{r} - 4r^4 \cdot r_i r_j}{r^{10}} = \frac{Q_{ij} r_i r_j}{2\pi\varepsilon_0 r^6}$$

$$\bar{\bar{Q}} = \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{xy} & Q_{yy} & Q_{yz} \\ Q_{xz} & Q_{yz} & -(Q_{xx}+Q_{yy}) \end{pmatrix}$$

$$\oiint \vec{E}\, d\vec{S} = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \Big|_{r=R} r^2 \sin\theta \cdot \frac{Q_{ij} r_i r_j}{2\pi\varepsilon_0 r^6} = \left| \begin{array}{l} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{array} \right| =$$

$$= \frac{1}{2\pi\varepsilon_0 R^2}\int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \Big[\underbrace{Q_{xx}\sin^3\theta\cos^2\varphi}_{I_{xx}} + \underbrace{2Q_{xy}\sin^3\theta\sin\varphi\cos\varphi}_{I_{xy}} +$$

$$+ \underbrace{2Q_{xz}\sin^2\theta\cos\theta\cos\varphi}_{I_{xz}} + \underbrace{Q_{yy}\sin^3\theta\sin^2\varphi}_{I_{yy}} + \underbrace{2Q_{yz}\sin^2\theta\cos\theta\sin\varphi}_{I_{yz}} +$$

$$+ \underbrace{Q_{zz}\sin\theta\cos^2\theta}_{I_{zz}} \Big]$$

$$\int_0^{2\pi} \sin\varphi\, d\varphi = \int_0^{2\pi}\cos\varphi\, d\varphi = 0 \Longrightarrow I_{xz} = I_{yz} = 0$$

$$\int_0^{2\pi} 2\sin\varphi\cos\varphi\, d\varphi = \int_0^{2\pi}\sin(2\varphi)\, d\varphi = 0 \Longrightarrow I_{xy} = 0$$

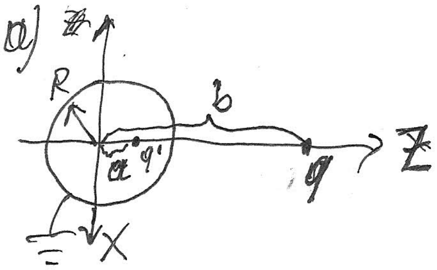
$$\int_0^{2\pi}\cos^2\varphi\, d\varphi = \int_0^{2\pi}\sin^2\varphi\, d\varphi = \pi \quad \left(= \int_0^{2\pi}\frac{1 \pm \cos 2\varphi}{2}\, d\varphi\right)$$

$$\oiint \vec{E}\, d\vec{S} = \frac{\pi}{2\pi\varepsilon_0 R^2}\int_0^{\pi} d\theta \left[(Q_{xx}+Q_{yy})\sin\theta(1-\cos^2\theta) - 2(Q_{xx}+Q_{yy})\sin\theta\cos^2\theta \right] =$$

$$= \left| \begin{array}{l} w = \cos\theta \\ dw = -\sin\theta\, d\theta \end{array} \right| = \frac{Q_{xx}+Q_{yy}}{2\varepsilon_0 R^2}\int_{-1}^{1} dw\,[1 - 3w^2] = \frac{Q_{xx}+Q_{yy}}{2\varepsilon_0 R^2}\left[w - w^3\right]_{-1}^{1} = \boxed{0}$$

yeah

6) a)



Uziemiona kula $\Rightarrow U(r=R)=0$

Załóżmy, że możemy użyć metody obrazów wykorzystując 1 ładunek

$a<R<b$

$U(r=R) \to \vec{r} = (R\sin\theta\cos\varphi,\ R\sin\theta\sin\varphi,\ R\cos\theta)$

$\vec{r}_A = (0,0,a)$; $\vec{r}_b = (0,0,b)$; $|r-r_a| = \sqrt{R^2+a^2-2Ra\cdot\cos\theta}$

$|r-r_b| = \sqrt{R^2+b^2-2Rb\cdot\cos\theta}$

$$U_{(r=R)} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{R^2+b^2-2Rb\cdot\cos\theta}} + \frac{q'}{\sqrt{R^2+a^2-2Ra\cdot\cos\theta}}\right] = 0$$

$$q\sqrt{R^2+a^2-2Ra\cos\theta} = -q'\sqrt{R^2+b^2-2Rb\cdot\cos\theta}$$

$$q^2R^2+q^2a^2-2q^2Ra\cdot\cos\theta = q'^2R^2+q'^2b^2-2q'^2Rb\cdot\cos\theta$$

$$2R(q'^2b-q^2a)\cdot\cos\theta + (q^2R^2+q^2a^2-q'^2R^2-q'^2b^2)=0$$

Aby $A\cdot\cos\theta + B = 0$ było dla każdego $\theta \to A=B=0$

$$\begin{cases} q'^2b-q^2a=0 \\ q^2R^2+q^2a^2-q'^2R^2-q'^2b^2 \end{cases} \to \begin{cases} q'^2 = \frac{a}{b}q^2 \\ q^2R^2+q^2a^2-\frac{a}{b}q^2R^2-\cancel{q^2b^2\frac{a}{b}}=0 \end{cases}$$

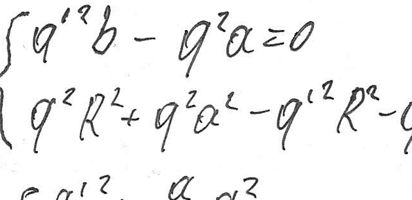
$$\begin{cases} q'^2 = \frac{a}{b}q^2 \\ q^2(R^2-\frac{a}{b}R^2+a^2-ab)=0 \end{cases} \to \begin{cases} -q' = \sqrt{\frac{a}{b}}\,q \\ a^2b-ab^2-aR^2+bR^2=0 \end{cases}$$

$$\Delta = (b^2+R^2)^2-4b^2R^2 = (b^2-R^2)^2$$

$$a = \frac{(b^2+R^2)\pm|b^2-R^2|}{2b} \xrightarrow{a<R<b} \frac{R^2}{b} \leftrightarrow \sqrt{\frac{a}{b}} = \sqrt{\frac{R^2}{b^2}} = \frac{R}{b}$$

$$\boxed{a\cdot b = R^2,\ bq'+Rq=0} \quad \left| a=\frac{R^2}{b},\ q'=-\frac{R}{b}q \right|$$

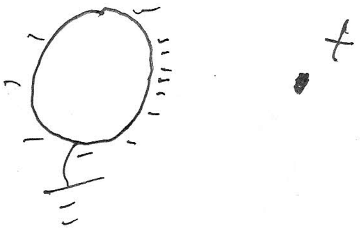
6 b)



Ładunek q'' daje potencjał $U'' = \frac{1}{4\pi\varepsilon_0}\frac{q''}{r}$, który jest stały na pow. sfery

Odpowiedź: [q' jak dla uziemionej, $q'+q''=Q$]

W pobliżu przewodnika $\vec{E} = \frac{\sigma}{\varepsilon_0}$



$$\sigma = \varepsilon_0 \vec{E}_{ind} = -\varepsilon_0 \vec{\nabla} U_{ind} = +\frac{q'}{4\pi}\frac{R - a\cdot\cos\theta}{(R^2+a^2-2Ra\cdot\cos\theta)^{3/2}}$$

uwaga! tylko od q'!

~~od ładunku przeciwnego!~~

można sprawdzić (żmudnym rachunkiem) że $\iint\limits_{sfera} \sigma\, ds = +q'$

$$\iint \sigma\, ds = \frac{+q'}{4\pi}\cdot\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\; R^2 \sin\theta \,\frac{R - a\cdot\cos\theta}{(R^2+a^2-2Ra\cdot\cos\theta)^{3/2}} = \frac{q'R^2}{2}\int_0^{\pi} d\theta\, \sin\theta\, \frac{R-a\cos\theta}{(R^2+a^2-2Ra\cos\theta)^{3/2}}$$

$$= \left|\begin{matrix} w = \cos\theta \\ dw = -\sin\theta\, d\theta \end{matrix}\right| = \frac{+q'R^2}{2}\int_{-1}^{1} dw\, \frac{R - aw}{(R^2+a^2-2Ra\cdot w)^{3/2}}$$

$$\int dw \frac{1}{(R^2+a^2-2Raw)^{3/2}} = \frac{1}{Ra}\frac{1}{\sqrt{R^2+a^2-2Raw}}\ ;\ \int dw \frac{w}{(R^2+a^2-2Raw)^{3/2}} = \frac{1}{Ra}\int dw\, w\cdot\left[\frac{1}{\sqrt{\cdots}}\right]' =$$

$$= \frac{1}{Ra}\frac{w}{\sqrt{R^2+a^2-2Ra\cdot w}} + \frac{1}{R^2a^2}\sqrt{R^2+a^2-2Ra\cdot w}$$

$$\iint \sigma\, ds = \frac{q'R^2}{2}\left[\frac{1}{a}\frac{1}{\sqrt{R^2+a^2-2Raw}} - \frac{w}{R}\frac{1}{\sqrt{R^2+a^2-2Raw}} - \frac{1}{R^2 a}\sqrt{R^2+a^2-2Raw}\right]_{-1}^{1} =$$

$$= \frac{q'R^2}{2}\left[\frac{1}{a(R-a)} - \frac{1}{R(R-a)} - \frac{R-a}{a^2R^2} - \frac{1}{a(R+a)} + \frac{-1}{R(R+a)} + \frac{R+a}{a^2R^2}\right] =$$

$$\cancel{= \frac{qR^2}{2R^2a^2(R^2-a^2)}\left[aR(R+a) - a^2(R+a) - (R-a)(R^2-a^2) - aR(R-a) + a^2(R-a) + (R+a)(R^2-a^2)\right]}$$

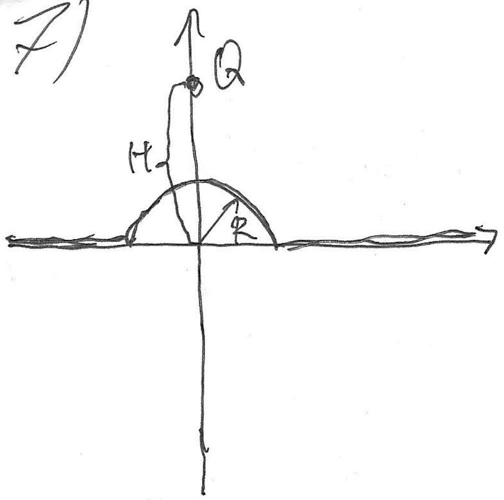
$$\cancel{= \frac{q}{2a^2(R^2-a^2)}\left[aR^2 + a^2R - a^2R - a^3 - R(R^2-a^2) + a(R^2-a^2) - aR^2 + a^2R + a^2R - a^3 + R(R^2-a^2) + a(R^2-a^2)\right]}$$

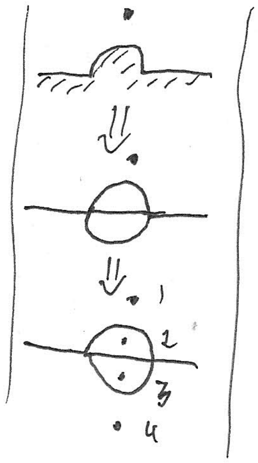
$$\cancel{= \frac{q}{2a^2(R^2-a^2)}\left[2a(R^2-a^2) + 2Ra^2 - 2a^3\right]}$$

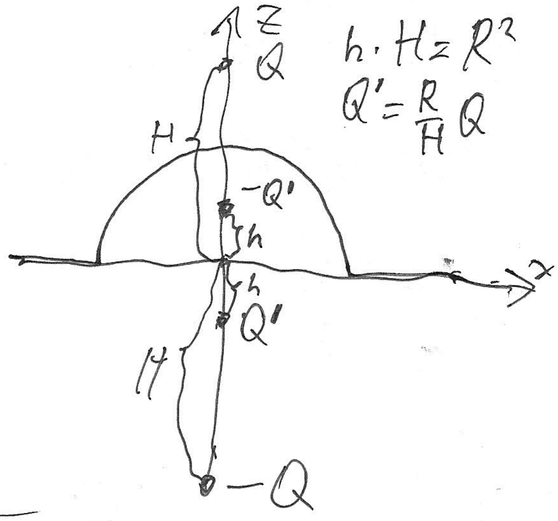
$$= \frac{q'}{2a(R^2-a^2)}\left[R^3 + R^2a - aR^2 - a^2R - R(R^2-a^2) + a(R^2-a^2) - R^3 + R^2a + aR^2 + a^2R + R(R^2-a^2) + a(R^2-a^2)\right]$$

$$= \frac{q'}{2a(R^2-a^2)}\left[2a(R^2-a^2)\right] = q'$$

7)







$h \cdot H = R^2$

$Q' = \frac{R}{H} Q$

$$\vec{F} = F_{12} + F_{13} + F_{14} = \boxed{\frac{Q}{4\pi\varepsilon_0}\left[\frac{-Q'}{(H-h)^2} + \frac{Q'}{(H+h)^2} - \frac{Q}{4H^2}\right]}$$

$$U_{ind} = \frac{1}{4\pi\varepsilon_0}\left[\frac{-Q'}{|r-r_2|} + \frac{Q'}{|r-r_3|} - \frac{Q}{|r-r_4|}\right]$$

$|r-r_2| = \sqrt{R^2+h^2-2Rh\cdot\cos\theta}$

$|r-r_3| = \sqrt{R^2+h^2+2Rh\cos\theta}$

$|r-r_4| = \sqrt{R^2+H^2+2RH\cos\theta}$

$$\sigma = \varepsilon_0 E_{ind} = -\varepsilon_0 \nabla U_{ind} = \begin{cases} -\varepsilon_0 \frac{\partial}{\partial z} U_{ind}\Big|_{z=0} & \text{płaszczyzna} \\ -\varepsilon_0 \frac{\partial}{\partial r} U_{ind}\Big|_{r=R} & \text{półsfera} \end{cases}$$

nie liczę bo to na 3-4 kartki

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1) $\bigcirc^{+Q}_{C_1}$ $\quad \bigcirc^{-Q}_{C_2}$ $\quad V_1 = \frac{Q}{C_1}$, $V_2 = \frac{-Q}{C_2} \to V = V_1 - V_2 = Q\frac{C_1+C_2}{C_1C_2}$ **Seria 4**

$$\boxed{C = \frac{Q}{V} = \frac{C_1C_2}{C_1+C_2}}$$

$\downarrow$

$\bigcirc$——$\bigcirc$ $\quad Q_1 + Q_2 = const \to C_1V_1 + C_2V_2 = (C_1+C_2)V \to V = \frac{C_1V_1+C_2V_2}{C_1+C_2}$

$$W = \sum \tfrac{1}{2} C_i V_i^2 \to W_{in} = \tfrac{1}{2}(C_1V_1^2 + C_2V_2^2),\ W_{out} = \tfrac{1}{2}(C_1+C_2)\left(\frac{C_1V_1+C_2V_2}{C_1+C_2}\right)^2 = \frac{1}{2}\frac{(C_1V_1+C_2V_2)^2}{C_1+C_2}$$

$$\Delta W = W_{out} - W_{in} = \frac{\cancel{C_1^2V_1^2} + \cancel{C_2^2V_2^2} + 2C_1C_2V_1V_2 - \cancel{C_1^2V_1^2} - C_1C_2V_1^2 - C_2C_1V_2^2 - \cancel{C_2^2V_2^2}}{2(C_1+C_2)} =$$

$$= -\frac{C_1C_2(V_1-V_2)^2}{2(C_1+C_2)} = \boxed{\frac{1}{2}C(V_1-V_2)^2}$$

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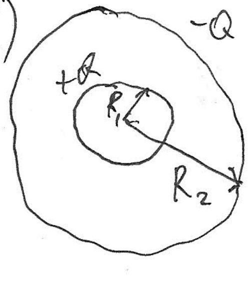
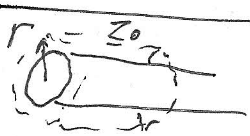
2)



$\frac{Q}{z_0} = 2\pi R_1 \sigma_+ = 2\pi R_2 \sigma_-$; z Gaussa



$$2\pi r E z_0 = \frac{Q}{\varepsilon_0} = \frac{2\pi R_1 \sigma_+ z_0}{\varepsilon_0} \to \vec{E} = \hat{e}_\rho \frac{R_1\sigma_+}{\varepsilon_0 r} \to U = -\frac{R_1\sigma_+}{\varepsilon_0}\ln\frac{r}{r_0}$$

$$\Delta U = U_1 - U_2 = \frac{R_1\sigma_+}{\varepsilon_0}\ln\frac{R_2}{R_1} \rightsquigarrow \frac{C}{z_0} = \frac{Q}{z_0\Delta U} = \boxed{\frac{2\pi\varepsilon_0}{\ln R_2/R_1}}$$