1a) $\varepsilon_{ijk}\varepsilon_{mjk} = \varepsilon_{kij}\varepsilon_{kmj} = \delta_{im}\overset{=3!!}{\delta_{jj}} - \delta_{ij}\delta_{mj} = 3\delta_{im} - \delta_{im} = 2\delta_{im}$

1b) $\varepsilon_{ijk}\varepsilon_{ijk} = \delta_{jj}\delta_{kk} - \delta_{jk}^2 = 9-3=6$ $\quad$ seria 1

2a) $\vec{a}\times(\vec{b}\times\vec{c}) = \varepsilon_{ijk}\hat{e}_i a_j(\varepsilon_{klm} b_l c_m) = \varepsilon_{kij}\varepsilon_{klm}\hat{e}_i a_j b_l c_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\cdot$
$\cdot\hat{e}_i a_j b_l c_m = \hat{e}_i a_j b_i c_j - \hat{e}_i a_j b_j c_i = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

2b) $(\vec{a}\times\vec{b})\times\vec{c} = \varepsilon_{ijk}\hat{e}_i(\varepsilon_{jlm}a_l b_m)c_k = \varepsilon_{jki}\varepsilon_{jlm}\hat{e}_i a_l b_m c_k =$
$= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})\hat{e}_i a_l b_m c_k = \hat{e}_i a_k b_i c_k - \hat{e}_i a_i b_k c_k = \vec{b}(\vec{a}\cdot\vec{c}) -$
$-\vec{a}(\vec{c}\cdot\vec{b})$

2c) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\varepsilon_{ijk}\hat{e}_i a_j b_k)\cdot(\varepsilon_{ilm}\hat{e}_i c_l d_m) = \varepsilon_{ijk}\varepsilon_{ilm} a_j b_k c_l d_m =$
$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) a_j b_k c_l d_m = a_j b_k c_j d_k - a_j b_k c_k d_j = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) -$
$-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

3) $\vec{\nabla}f(r) = \hat{e}_i\,\partial_i f(r) = \hat{e}_i\,\partial_r f\cdot\partial_i r = \hat{e}_i f'(r)\partial_i\sqrt{r_j r_j} =$ $\quad$ $\boxed{\frac{\partial}{\partial r_i} \equiv \partial_i}$
$= \hat{e}_i f'\cdot\frac{1}{2\sqrt{r_j r_j}}\partial_i(r_k^2) = \frac{\hat{e}_i f'}{2|r|}\cdot 2r_k\delta_{ik} = \frac{\hat{e}_i r_i}{|r|}f' = f'\frac{\vec{r}}{|r|} = f'\vec{e}_r$
$\vec{\nabla}r = \hat{e}_i\frac{r_i}{r}$

4) $\vec{n} = const \rightarrow \partial_i n_j = 0;\quad \partial_i r_j = \delta_{ij}$ $\quad$ $\boxed{|r| \equiv r \neq \vec{r}}$

4a) $\vec{\nabla}(\vec{n}\cdot\vec{r}) = \hat{e}_i\partial_i n_j r_j = \hat{e}_i n_j\delta_{ij} = \hat{e}_i n_i = \vec{n}$

4b) $(\vec{n}\cdot\vec{\nabla})\vec{r} = \hat{e}_i n_j\partial_j r_i = \hat{e}_i n_j\delta_{ij} = \hat{e}_i n_i = \vec{n}$ $\quad$ woow =.=

4c) $\vec{\nabla}\frac{\vec{n}\cdot\vec{r}}{r^3} = \hat{e}_i\partial_i\frac{n_j r_j}{(r_k)^{3/2}} = \partial_i\left(\frac{n_j r_j}{r^3}\right) = \hat{e}_i\frac{n_j\delta_{ij}r^3 - n_j r_j\cdot 3r^2\frac{r_i}{r}}{r^6} =$ $\quad$ $\boxed{\partial_i r = \frac{r_i}{r}}$

$$= \frac{n_i\hat{e}_i}{r^3} - 3\frac{\hat{e}_i n_j r_j r_i}{r^5} = \frac{\vec{n}}{r^3} - \frac{(\vec{n}\cdot\vec{r})\vec{r}}{r^5}$$

5a) $\vec{\nabla}\cdot(f\vec{G}) = \hat{e}_i\partial_i(fG_i) = \hat{e}_i f\partial_i G_i + \hat{e}_i G_i\partial_i f = f(\vec{\nabla}\cdot\vec{G}) + \vec{G}\cdot\vec{\nabla}f$

5b) $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \hat{e}_i\partial_i\,\varepsilon_{ijk}\hat{e}_i A_j B_k = \cancel{\varepsilon_{ijk}\hat{e}_j}$ to be continued...

5b) $\varepsilon_{ijk}\partial_i A_j B_k = \varepsilon_{ijk} A_j \partial_i B_k + \varepsilon_{ijk} B_k \partial_i A_j = (\varepsilon_{kij}\hat{e}_k \partial_i A_j)\hat{e}_k B_k -$

$- (\varepsilon_{jik}\hat{e}_j \partial_i B_k)\hat{e}_j A_j = \vec{B}\cdot(\vec\nabla\times\vec A) - \vec A\cdot(\vec\nabla\times\vec B)$

5c) $\vec\nabla\times(\vec\nabla\times\vec u) = \varepsilon_{ijk}\hat e_i \partial_j(\varepsilon_{klm}\partial_l u_m) = \varepsilon_{kij}\varepsilon_{klm}\hat e_i\partial_j\partial_l u_m =$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\hat e_i \partial_j\partial_l u_m = \hat e_i \partial_j\partial_i u_j - \hat e_i \partial_j^2 u_i = \vec\nabla(\vec\nabla\cdot\vec u) - \Delta\vec u$

6a) $\vec F = (\vec A\cdot\vec r)\vec B = A_j r_j B_i \hat e_i$ $\quad\boxed{\partial_i A_j = \partial_i B_j = 0,\ \partial_i r_j = \delta_{ij}}$

$\vec\nabla\cdot\vec F = \partial_i A_j B_i r_j = A_j B_i \partial_i r_j = A_i B_i = \vec A\cdot\vec B$

$\nabla\times\vec F = \varepsilon_{ijk}\hat e_i \partial_j (A_l r_l B_k) = \varepsilon_{ijk}\hat e_i A_l B_k \partial_j r_l = \varepsilon_{ijk}\hat e_i A_j B_k =$

$= \vec A\times\vec B$

6b) $\vec F = \frac12(\vec B\times\vec r)$

$\vec\nabla\cdot\vec F = \frac12\partial_i\varepsilon_{ijk}B_j r_k = \frac12 B_j\varepsilon_{iji} = 0$

$\vec\nabla\times\vec F = \varepsilon_{ijk}\hat e_i\partial_j\left(\frac12\varepsilon_{klm}B_l r_m\right) = \frac{\hat e_i}{2}\varepsilon_{kij}\varepsilon_{klm}\partial_j B_l r_m =$

$= \frac{\hat e_i}{2}(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})B_l\partial_j r_m = \frac{\hat e_i}{2}B_i\partial_j r_j - \frac{\hat e_i}{2}B_j\partial_j r_i =$

$= \frac{\hat e_i}{2}(3B_i - \delta_{ij}B_j) = \hat e_i B_i = \vec B$

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7) goto (Electro Vademecum) | 8–10) tu nie zmieszczę się

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11) Jeżeli $\oint\varphi\, d\vec r = \iint \vec n\times\vec\nabla\varphi\, dS$ to $\vec C\cdot\oint\varphi\,d\vec r = \vec C\cdot\iint\vec n\times\vec\nabla\varphi\,dS$ i odwrotnie

$\boxed{\vec C = const}$ $\boxed{\vec\varphi = \varphi\cdot\vec C}$

$\vec C\cdot\oint\varphi\,d\vec r = \oint\vec\varphi\cdot d\vec r = \iint(\vec\nabla\times\vec\varphi)\cdot d\vec S = \iint(\vec\nabla\times\varphi\vec C)\cdot\vec n\,dS = \iint\varepsilon_{ijk}\hat e_i\partial_j(\varphi C_k)\cdot$

$\cdot n_i\hat e_i\,dS = \iint(\varepsilon_{kij}\hat e_k n_i\partial_j\varphi)\cdot C_k\hat e_k\,dS = \iint\vec C\cdot(\vec n\times\vec\nabla\varphi)dS = \vec C\cdot\iint\vec n\times\vec\nabla\varphi\,dS$

c.k.d.

8) $x = r\cdot\sin\theta\cdot\cos\varphi, \; y = r\cdot\sin\theta\cdot\sin\varphi, \; z = r\cos\theta$

$$h_r^2 = (\partial_r x)^2 + (\partial_r y)^2 + (\partial_r z)^2 = \sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta = 1$$

$$\boxed{h_r = 1}$$

$$h_\theta^2 = (\partial_\theta x)^2 + (\partial_\theta y)^2 + (\partial_\theta z)^2 = (r\cos\theta\cos\varphi)^2 + (r\cos\theta\sin\varphi)^2 + (-r\sin\theta)^2 = r^2$$

$$\boxed{h_\theta = r}$$

$$h_\varphi^2 = (-r\sin\theta\sin\varphi)^2 + (r\sin\theta\cos\varphi)^2 = r^2\sin^2\theta \qquad \boxed{h_\varphi = r\sin\theta}$$

$$\hat{e}_r = \frac{\partial_r \vec{r}}{h_r} = \begin{pmatrix} \sin\theta\cdot\cos\varphi \\ \sin\theta\cdot\sin\varphi \\ \cos\theta \end{pmatrix}, \quad \hat{e}_\theta = \frac{\partial_\theta \vec{r}}{h_\theta} = \begin{pmatrix} \cos\theta\cdot\cos\varphi \\ \cos\theta\cdot\sin\varphi \\ -\sin\theta \end{pmatrix}, \quad \hat{e}_\varphi = \frac{\partial_\varphi \vec{r}}{h_\varphi} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$d\vec{r} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + r\sin\theta\,d\varphi\,\hat{e}_\varphi$$

$$d\vec{S} = r^2\sin\theta\,d\theta\,d\varphi\,\hat{e}_r + r\sin\theta\,dr\,d\varphi\,\hat{e}_\theta + r\,dr\,d\theta\,\hat{e}_\varphi$$

$$dV = r^2\sin\theta\,dr\,d\theta\,d\varphi$$

9) $$\vec{\nabla} f = \hat{e}_r\,\partial_r f + \hat{e}_\theta \frac{1}{r}\partial_\theta f + \hat{e}_\varphi \frac{1}{r\sin\theta}\partial_\varphi f$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2\sin\theta}\left(\partial_r r^2\sin\theta A_r + \partial_\theta r\sin\theta A_\theta + \partial_\varphi r A_\varphi\right) =$$

$$= \frac{1}{r^2}\partial_r(r^2 A_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\partial_\varphi A_\varphi$$

$$\vec{\nabla}\times\vec{A} = \frac{\hat{e}_r}{r^2\sin\theta}\left(\partial_\theta r\sin\theta A_\varphi - \partial_\varphi r A_\theta\right) + \frac{\hat{e}_\theta}{r\sin\theta}\left(\partial_\varphi A_r - \partial_r r\sin\theta A_\varphi\right) +$$

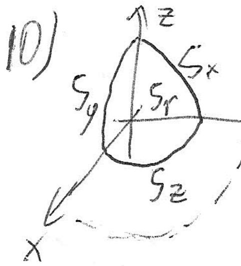
$$+ \frac{\hat{e}_\varphi}{r}\left(\partial_r r A_\theta - \partial_\theta A_r\right) = \frac{\hat{e}_r}{r\cdot\sin\theta}\left(\partial_\theta(\sin\theta A_\varphi) - \partial_\varphi A_\theta\right) + \frac{\hat{e}_\theta}{r}\left(\frac{\partial_\varphi A_r}{\sin\theta} - \partial_r r A_\varphi\right)$$

$$+ \frac{\hat{e}_\varphi}{r}\left(\partial_r(r A_\theta) - \partial_\theta A_r\right)$$

$$\Delta f = \frac{1}{r^2\sin\theta}\left(\partial_r r^2\sin\theta\,\partial_r f + \partial_\theta \frac{r\sin\theta}{r}\partial_\theta f + \partial_\varphi \frac{r}{r\sin\theta}\partial_\varphi f\right) =$$

$$= \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta f) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 f$$

10)



$V: r\in[0,R],\ \theta\in[0,\frac{\pi}{2}],\ \varphi\in[0,\frac{\pi}{2}]$

powierzchnie: $S_r \to r=R$

$S_y \to \varphi = 0$

$S_x \to \varphi = \pi/2$

$S_z \to \theta = \frac{\pi}{2}$

$$\vec{u} = \begin{pmatrix} r^2\cos\theta \\ r^2\cos\varphi \\ -r^2\cos\theta\sin\varphi \end{pmatrix}$$

(w $\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi$)

UWAGA! $\hat{e}_\varphi$ wchodzi wewnątrz przez S_y – tam strumień ma przeciwny znak!

Gauss: $\iiint \vec{\nabla}\cdot\vec{u}\, dV = \oiint \vec{u}\cdot d\vec{s}$

$$\vec{\nabla}\cdot\vec{u} = \frac{1}{r^2}\partial_r(r^4\cos\theta) + \frac{1}{r\sin\theta}\partial_\theta(r^2\cos\varphi\sin\theta) + \frac{1}{r\sin\theta}\partial_\varphi(-r^2\cos\theta\sin\varphi) =$$

$$= 4r\cos\theta + r\cos\varphi\,\mathrm{ctg}\,\theta - r\cos\varphi\,\mathrm{ctg}\,\theta = \boxed{4r\cos\theta}$$

$$\iiint \vec{\nabla}\cdot\vec{u}\, dV = \int_0^R dr \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\varphi\, r^2\sin\theta\cdot 4r\cos\theta = \int_0^R 4r^3 dr \int_0^{\pi/2} \frac{1}{2}\sin(2\theta)\, d\theta \int_0^{\pi/2} d\varphi =$$

$$= [r^4]_0^R \cdot \left[-\frac{1}{4}\cos(2\theta)\right]_0^{\pi/2} \cdot \pi/2 = \boxed{\frac{\pi R^4}{4}}$$

$$S_r: \vec{n} = \hat{e}_r \to \iint_{S_r} \vec{u}\cdot\vec{n}\, ds = \Big|_{r=R} \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\varphi\, r^2\sin\theta\cdot u_r = \int_0^{\pi/2} R^4\sin\theta\cos\theta\, d\theta \int_0^{\pi/2} d\varphi =$$

$$= \frac{\pi R^4}{2}\cdot\int_0^{\pi/2}\frac{1}{2}\sin(2\theta)\, d\theta = \text{(patrz wyżej)}\ \boxed{\frac{\pi R^4}{4}}$$

$$S_z: \vec{n} = \hat{e}_\theta \to \iint_{S_z} \vec{u}\cdot\vec{n}\, ds = \Big|_{\theta=\frac{\pi}{2}} \int_0^R dr \int_0^{\pi/2} d\varphi\, r\sin\theta\, u_\theta = \int_0^R r^3 dr \int_0^{\pi/2} \cos\varphi\, d\varphi = \boxed{\frac{R^4}{4}}$$

$$S_x: \vec{n} = \hat{e}_\varphi \to \iint_{S_x} \vec{u}\cdot\vec{n}\, ds = \Big|_{\varphi=\frac{\pi}{2}} \int_0^R dr \int_0^{\pi/2} d\theta\, r\, u_\varphi = -\int_0^R r^3 dr \int_0^{\pi/2} \cos\theta\, d\theta\cdot \sin\frac{\pi}{2} = \boxed{-\frac{R^4}{4}}$$

$$S_y: \vec{n} = -\hat{e}_\varphi \to \iint_{S_y} \vec{u}\cdot\vec{n}\, ds = \Big|_{\varphi=0} \int_0^R dr \int_0^{\pi/2} d\theta\, (-r\, u_\varphi) = \int_0^R r^3 dr \int_0^{\pi/2} \cos\theta\, d\theta\cdot \underbrace{\sin 0}_{=0} = \boxed{0}$$

$$\oiint_{\partial V} \vec{u}\, d\vec{s} = \iint_{S_r} + \iint_{S_x} + \iint_{S_y} + \iint_{S_z} = \frac{\pi R^4}{4} + \frac{R^4}{4} - \frac{R^4}{4} + 0 = \boxed{\frac{\pi R^4}{4}}$$

nailed it