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A Simple Proof of the Robinson Theorem

by

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1. – Notation

Standard terminology and notation of algebra and differential geometry is used in this paper. The exterior algebra of an n-dimensional real vector space V is denoted by

$$\Lambda V^* = \bigoplus_{k=0}^{n} \Lambda^k V^*,$$

where $\Lambda^0 V^* = \mathbb{R}$ and $\Lambda^1 V^* = V^*$ is the dual of V. If $u \in V$, then

$$i(u) : \Lambda V^* \to \Lambda V^*$$

is the (anti) derivation of degree -1 defined by

$$i(u)\alpha = \langle u, \alpha \rangle \qquad \text{for any } \alpha \in V^*$$

and

$$i(u)(\beta \wedge \gamma) = (i(u)\beta) \wedge \gamma + (-1)^k \beta \wedge i(u)\gamma$$

for any $\beta \in \Lambda^k V^*$. Sometimes one writes $u \lrcorner \alpha$ instead of $i(u)\alpha$. If

$$A : V \to V$$

is a linear map, then

$$\bar{A} : \Lambda V^* \to \Lambda V^*$$

denotes the derivation of degree 0 defined by

$$\langle u, \bar{A}\alpha \rangle = \langle A u, \alpha \rangle \qquad \text{for any } u \in V \quad \text{and } \alpha \in V^*$$

and

$$\bar{A}(\beta \wedge \gamma) = (\bar{A}\beta) \wedge \gamma + \beta \wedge \bar{A}\gamma, \qquad \beta, \gamma \in \Lambda V^*.$$

It is easy to check that

(1) $$[i(u), \bar{A}] = i(Au)$$

for any $u \in V$ and $A \in \mathscr{L}(V)$. If $u \in V$ and $\alpha \in V^*$ then the map A defined by $Av = \alpha(v)u$ is written as $A = u \otimes \alpha$.

A scalar product in V is defined as a bilinear symmetric map $g : V \times V \to \mathbb{R}$ which is non-degenerate, but the quadratic form $u \to g(u, u)$ needs not be positive-definite. The same letter g will be used to denote the isomorphism of V onto V^* defined by

$$\langle v, g(u) \rangle = g(u, v), \qquad u, v \in V.$$

A linear map $A : V \to V$ is symmetric with respect to g if, for any $u, v \in V$,

$$g(Au, v) = g(u, Av).$$

If A is symmetric, then $g(Au) = \bar{A}g(u)$.

Let (e_μ), $\mu = 1, \ldots, n$, be a linear frame (basis) in V and let (e^μ) denote its dual:

$$\langle e_\mu, e^\nu \rangle = \delta^\nu_\mu.$$

The n-form

(2) $$e = e^1 \wedge e^2 \wedge \ldots \wedge e^n$$

spans $\Lambda^n V^*$ and

(3) $$\bar{A}e = e\, Tr A.$$

Assume now that V has a preferred orientation and consider a frame which agrees with the orientation and is unimodular, i.e.:

$$|\det (g_{\mu\nu})| = 1,$$

where

$$g_{\mu\nu} = g(e_\mu, e_\nu).$$

The n-form (2) is now called an (oriented) volume element. The Hodge dual is an isomorphism of the vector space ΛV^* on itself,

$$* : \Lambda V^* \to \Lambda V^*,$$

defined as follows. Let $\alpha \in \Lambda^k V^*$ and $u_{k+1}, \ldots, u_n \in V$, then $* \alpha \in \Lambda^{n-k} V^*$ is given by

(4) $$* \alpha(u_{k+1}, \ldots, u_n)\, e = \alpha \wedge g(u_{k+1}) \wedge \ldots \wedge g(u_n).$$

One has

$$(5) \qquad i(u) * \alpha = *(\alpha \wedge g(u))$$

and, if $A \in \mathscr{L}(V)$ is symmetric,

$$(6) \qquad \bar{A} * + * \bar{A} = (TrA)*.$$

Let M be an n-dimensional smooth oriented manifold with a metric tensor g. The algebraic notions and constructions described above are extended, in a natural manner, to smooth fields on M. For example, if $\Gamma(M) = \oplus \Gamma^k(M)$ is the Cartan algebra of differential forms on M and u is a vector field, then $i(u) : \Gamma(M) \to \Gamma(M)$ is a derivation of degree -1. The exterior derivative

$$\mathrm{d} : \Gamma(M) \to \Gamma(M)$$

is a derivation of degree $+1$. If u and v are vector fields, then

$$(7) \qquad \mathscr{L}(u) = \mathrm{d} \circ i(u) + i(u) \circ \mathrm{d}$$

is a derivation of degree 0 (the Lie derivative with respect to u); we have:

$$(8) \qquad [\mathscr{L}(u), \mathrm{d}] = 0$$

and

$$(9) \qquad [\mathscr{L}(u), i(v)] = i([u, v]),$$

where $[u, v]$ is the usual bracket of vector fields,

$$(10) \qquad \mathscr{L}([u, v]) = [\mathscr{L}(u), \mathscr{L}(v)].$$

If $A : TM \to TM$ is an endomorphism of the tangent bundle TM, then $\bar{A}$ denotes the corresponding derivation of the Cartan algebra; there are obvious extensions of formulae (1) - (6) to fields on M. If u and v are vector fields and $g(u)$ denotes the 1-form corresponding to u under the isomorphism $g : TM \to T^*M$, then the map

$$v \longmapsto \mathscr{L}(u)(g(v)) - g([u, v])$$

defines a tensor field (the Lie derivative of g with respect to u),

$$\mathscr{L}_u g : TM \to T^*M,$$

given by:

$$(11) \qquad (\mathscr{L}_u g)(v) = \mathscr{L}(u)(g(v)) - g([u, v]).$$

This tensor field is symmetric, $\langle w, (\mathscr{L}_u g)(v)\rangle = \langle v, (\mathscr{L}_u g)(w)\rangle$, and it vanishes if and only if u generates a group of isometries of g. The composed map

(12) $$A_u = g^{-1} \circ \mathscr{L}_u g : TM \to TM$$

occurs in the following lemma.

LEMMA 1. *Let*

$$* : \Gamma(M) \to \Gamma(M)$$

be the Hodge dual acting on differential forms. If u is a vector field on M, then

(13) $$[\mathscr{L}(u), *] = (\overline{A}_u - (1/2)\, Tr\, A_u\, \overline{\mathrm{id}})\, *.$$

Moreover, according to (1), *there holds*

(14) $$[i(u), \overline{A}_u] = i(A_u u),$$

where

$$g(A_u u) = (\mathscr{L}_u g)(u)$$

and

$$Tr\, A_u = 2\, \mathrm{div}\, u.$$

It is also clear that $[\mathscr{L}(u), *]$ anticommutes with $*$ and

$$\overline{\mathrm{id}} \,|\, \Gamma^k(M) = k\, \mathrm{id} \,|\, \Gamma^k(M).$$

2. – Spacetime and the Maxwell Equations

A spacetime is a (space and time) orientable four-dimensional manifold M with a metric tensor g of signature -2. The Hodge dual acting on 2-forms is invariant under conformal changes of g. It depends only on the conformal geometry of M. As a result of this, Maxwell's equations in empty space,

(15) $$\mathrm{d}F = 0, \qquad \mathrm{d} * F = 0$$

where $F \in \Gamma^2(M)$, are conformally invariant.

Let k be a complete, nowhere vanishing vector field on M. There then exists a smooth map

$$\varphi : \mathbb{R} \times M \to M, \qquad \varphi(t, p) = \varphi_t(p),$$

such that

$$\varphi_t \circ \varphi_s = \varphi_{t+s}, \qquad \varphi_0 = \mathrm{id}_M$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi_t) = \mathscr{L}(k)(f \circ \varphi_t)$$

for any smooth function f. One says that (φ_t) is the flow generated by k. Assume that there exists a hypersurface $S \subset M$ transversal to k and that the restriction ψ of φ to $\mathbb{R} \times S$ is a diffeomorphism of $\mathbb{R} \times S$ onto M. A system of (local) coordinates (x', y', z') on S can be used to define coordinates (t, x, y, z) in (a suitable region of) M by putting (cf. fig. 1)

$$t = pr_1 \circ \psi^{-1}$$

and

$$x = x' \circ pr_2 \circ \psi^{-1}, \text{ etc.}$$

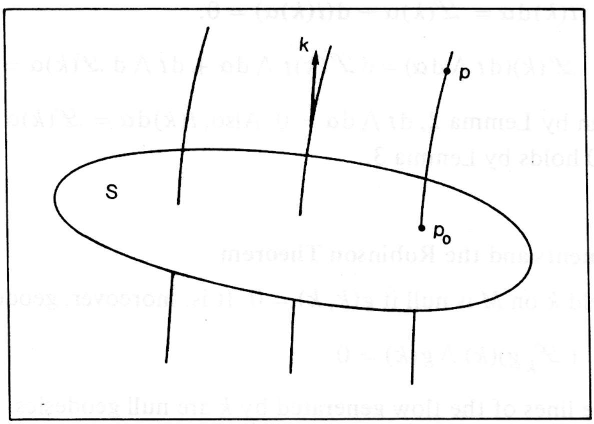

Fig. 1

It follows from the definition that $k = \partial/\partial t$ and

$$\langle k, \mathrm{d}t \rangle = 1, \qquad \langle k, \mathrm{d}x \rangle = 0, \quad \text{etc.}$$

$$p = \varphi_t(p_0), \qquad x(p) = x'(p_0), \quad \text{etc.}$$

The following lemmas are straightforward.

LEMMA 2. *Let $\alpha \in \Gamma(M)$ and B be an endomorphism of the tangent bundle of M. If*

$$\alpha \,|\, S = 0 \qquad \text{and} \qquad \mathscr{L}(k)\alpha = \overline{B}\alpha$$

then

$$\alpha = 0.$$

LEMMA 3. *If* $i(k)\alpha = 0$ *and* $\mathrm{d}t \wedge \alpha = 0$, *then* $\alpha = 0$.

LEMMA 4. *If*

$$\mathscr{L}(k)\alpha = 0, \qquad i(k)\alpha \,|\, S = 0 \qquad \text{and} \qquad (\mathrm{d}t \wedge \mathrm{d}\alpha) \,|\, S = 0$$

then

$$i(k)\alpha = 0 \qquad \text{and} \qquad \mathrm{d}\alpha = 0.$$

Indeed,

$$\mathscr{L}(k)(i(k)\alpha) = i(k)\ \mathscr{L}(k)\alpha = 0$$

implies $i(k)\alpha = 0$ (Lemma 2). Moreover,

$$i(k)\mathrm{d}\alpha = \mathscr{L}(k)\alpha - \mathrm{d}(i(k)\alpha) = 0,$$

$$\mathscr{L}(k)(\mathrm{d}t \wedge \mathrm{d}\alpha) = \mathrm{d}\mathscr{L}(k)t \wedge \mathrm{d}\alpha + \mathrm{d}t \wedge \mathrm{d}\ \mathscr{L}(k)\alpha = 0;$$

therefore, again by Lemma 2, $\mathrm{d}t \wedge \mathrm{d}\alpha = 0$. Also, $i(k)\mathrm{d}\alpha = \mathscr{L}(k)\alpha - \mathrm{d}(i(k)\alpha) = 0$ so that $\mathrm{d}\alpha = 0$ holds by Lemma 3. (Q.E.D.)

3. – Null Elements and the Robinson Theorem

A vector field k on M is null if $g(k, k) = 0$. It is, moreover, geodesic if

(16) $$(\mathscr{L}_k g)(k) \wedge g(k) = 0.$$

In this case the lines of the flow generated by k are null geodesics.

The form $\alpha \in \Gamma(M)$ is null if there exists a nowhere vanishing vector field k such that

$$i(k)\alpha = 0 \qquad \text{and} \qquad i(k) * \alpha = 0.$$

If $\alpha \neq 0$ then the vector field k is necessarily null (use (5) to prove this).

THEOREM 1. *Let* k *be null and geodesic. If*

$$\mathscr{L}(k)\alpha = 0 \qquad \text{and} \qquad i(k) * \alpha \,|\, S = 0$$

then $i(k) * \alpha = 0$.

Proof. Since

$$i(k) * \alpha = * (\alpha \wedge g(k)),$$

the theorem is equivalent to the following: if k is null geodesic, $\mathscr{L}(k)\alpha = 0$ and

$\alpha \wedge g(k) \,|\, S = 0$ then $\alpha \wedge g(k) = 0$. Now,

$$\mathscr{L}(k)(\alpha \wedge g(k)) = (\mathscr{L}(k)\alpha) \wedge g(k) + \alpha \wedge (\mathscr{L}_k g)(k) = \alpha \wedge (\mathscr{L}_k g)(k).$$

Since $(\mathscr{L}_k g)(k)$ is parallel to $g(k)$, the right-hand side of the last equation is proportional to $\alpha \wedge g(k)$ and Theorem 1 follows from Lemma 2. (Q.E.D.)

LEMMA 5. *Let $F \in \Gamma^2(M)$ be non-zero and null, $i(k)F = 0 = i(k) * F$, $k \neq 0$ and let B be a traceless, symmetric endomorphism of TM. Condition $\bar{B}F = 0$ is equivalent to the existence of a vector field u such that*

$$B = u \otimes g(k) + k \otimes g(u) - \frac{1}{2} g(u, k) \,\mathrm{id}. \tag{17}$$

A proof of the lemma is obtained by constructing a frame (e_μ) such that $g = g_{\mu\nu} e^\mu \otimes e^\nu = e^3 \otimes e^4 + e^4 \otimes e^3 - e^1 \otimes e^1 - e^2 \otimes e^2$ and $F = f e^1 \wedge e^3$, $*F = f e^2 \wedge e^3$. One writes $B = B^\mu{}_\nu e_\mu \otimes e^\nu$ where $B_{\mu\nu} = B_{\nu\mu} = g_{\mu\rho} B^\rho{}_\nu$ and $TrB = B^\rho{}_\rho = 0$. It follows from (6) that $\bar{B}$ anticommutes with $*$ so that $\bar{B} * F = 0$. The rest is a computation. (Q.E.D.)

THEOREM 2. *If $F \in \Gamma^2(M)$ is non-zero and null, $i(k)F = 0$,*

$$0 = i(k) * F, \qquad k \neq 0,$$

and

$$\mathscr{L}(k)F = 0, \qquad \mathscr{L}(k) * F = 0,$$

then there exists a vector field u such that

$$A_k - \frac{1}{4}(TrA_k)\mathrm{id} = u \otimes g(k) + k \otimes g(u) - \frac{1}{2} g(u, k)\mathrm{id}. \tag{18}$$

Proof. It follows from the assumptions of the theorem that $[\mathscr{L}(k), *]F = 0$. Lemmas 1 and 5 complete the proof. (Q.E.D.)

REMARK. Condition (18) is equivalent to the following: there exist a function a and a vector field u such that

$$\mathscr{L}_k g = 2ag + g(u) \otimes g(k) + g(k) \otimes g(u). \tag{19}$$

Clearly, if (19) is satisfied, then $(\mathscr{L}_k g)(k) \wedge g(k) = 0$ so that k is geodesic. It has been shown elsewhere [5] that the flow generated by k subject to (19) preserves the distribution of subspaces orthogonal to k, together with their (degenerate) conformal structure induced by g. For this reason, the flow, and k itself, is said to be null, geodesic and shearfree ([2]; cf. also [1], [3], [6]).

THEOREM 3. *Consider a null geodesic and shearfree, non-zero vector field k and a hypersurface S transversal to k and such that the flow generated by k determines a diffeomorphism of $\mathbb{R} \times S$ onto M. If $F \in \Gamma^2(M)$ satisfies the following initial conditions*

(20) $$i(k)F|S = 0, \qquad i(k) * F|S = 0,$$

(21) $$\mathrm{d}t \wedge \mathrm{d}F|S = 0, \qquad \mathrm{d}t \wedge \mathrm{d} * F|S = 0,$$

and is invariant by the flow,

$$\mathscr{L}(k)F = 0,$$

then F is a null solution of Maxwell's equations,

$$\mathrm{d}F = 0 \quad \textit{and} \quad \mathrm{d} * F = 0.$$

Proof. It follows from Lemma 4 that $i(k)F = 0$ and $\mathrm{d}F = 0$. Theorem 1 yields $i(k) * F = 0$ so that F is null. Since $\mathscr{L}(k) * F = [\mathscr{L}(k), *]F = 0$ by Lemmas 1 and 5, Lemma 4 can be applied to $\alpha = * F$ to get $\mathrm{d} * F = 0$. (Q.E.D.)

REMARK. The initial data (21) contain derivatives of F only in directions tangential to S. There always are non-zero initial data statisfying (20 - 21). This can be seen from the following argument [4]: let v be a unit vector field on S, tangent to S and orthogonal to k. Then $* (g(k) \wedge g(v)) = g(k) \wedge g(w)$ where w has unit length and is orthogonal to both k and v; it may be chosen to be tangent to S. Put $F|S = g(k) \wedge (ag(v) + bg(w))$ where a and b are functions on S. Conditions (20 - 21) reduce to two first order linear differential equations for a and b which may be solved.

COROLLARY *(The Robinson Theorem). With any null, geodesic and shearfree vector field k there is associated a non-trivial null solution of Maxwell's equations.*

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Abstract. *It is shown that if a 2-form F in a 4-dimensional conformal spacetime is invariant by the action of the flow generated by a null, geodesic and shearfree vector field k and satisfies the initial conditions:* $k \lrcorner F = 0 = k \lrcorner * F$ *and* $\mathrm{d}t \wedge \mathrm{d}F = 0 = \mathrm{d}t \wedge \mathrm{d} * F$ *on a hypersurface* $t = const.$ *transversal to k, then F is a null Maxwell field. The proof depends on a useful formula for the commutator of the Lie derivative with the Hodge * operator.*

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